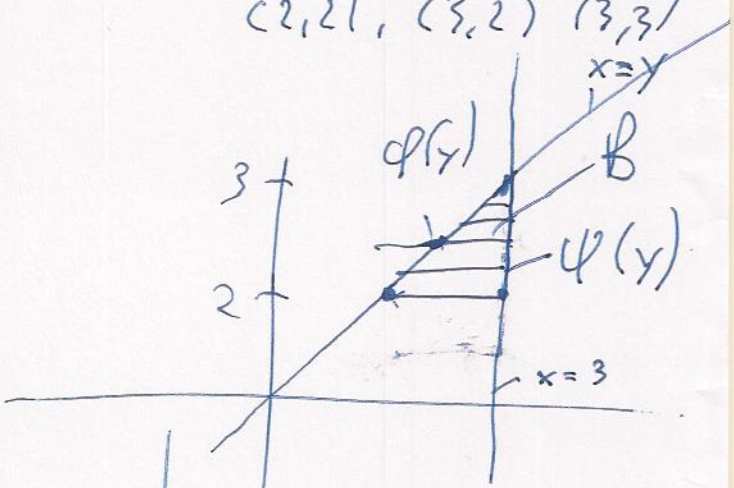
$$\iint_B \frac{x-y}{x+y}\, dx\, dy = \quad , \quad B \text{ durch } \Delta \text{ geg.:}$$

$(2,2), (3,2)\ (3,3)$



$\psi(y) = 3$

$\varphi(y) = y$

$$= \int_2^3 \left(\int_{\varphi(y)}^{\psi(y)} \frac{x-y}{x+y} \cdot dx \right) dy =$$

$$= \int_{y=2}^{3} \left(\int_{x=y}^{3} \frac{x-y}{x+y} \cdot dx \right) dy = \int_2^3 \int_y^3 \frac{x+y-2y}{x+y} \cdot dx\, dy =$$

$$= \int_2^3 \left[\int_y^3 \left(1 - \frac{2y}{x+y}\right) \cdot dx \right] \cdot dy =$$

$$= \int_2^3 \left(\left(x - 2y \cdot \ln(x+y) \right) \Big|_y^3 \right) \cdot dy =$$

$$= \int_2^3 \left(3 - 2y \cdot \ln(3+y) - y + 2y \cdot \ln(2y) \right) \cdot dy =$$

$$= \int_2^3 \left(3 - 2y \cdot \ln(3+y) - y + 2y \cdot \ln 2 + 2y \cdot \ln y \right) \cdot dy$$

$\Downarrow$ Stammfkt. berechnen $F(y)$

$$\int y \cdot \ln y \cdot dy = \frac{y^2}{2} \cdot \ln y - \int \frac{y^2}{2} \cdot \frac{1}{y} \cdot dy =$$

$$= \frac{y^2}{2} \cdot \ln y - \frac{1}{2} \int y \cdot dy =$$

$$= \frac{y^2}{2} \cdot \ln y - \frac{y^2}{4}$$

$$\int y \cdot \ln(3+y) \cdot dy = \int (3+y) \cdot \ln(3+y) dy -$$

$$- 3 \int \ln(3+y) \cdot dy = *$$

$$\int 1 \cdot \ln y \cdot dy = y \cdot \ln y - \int y \cdot \frac{1}{y} \cdot dy =$$

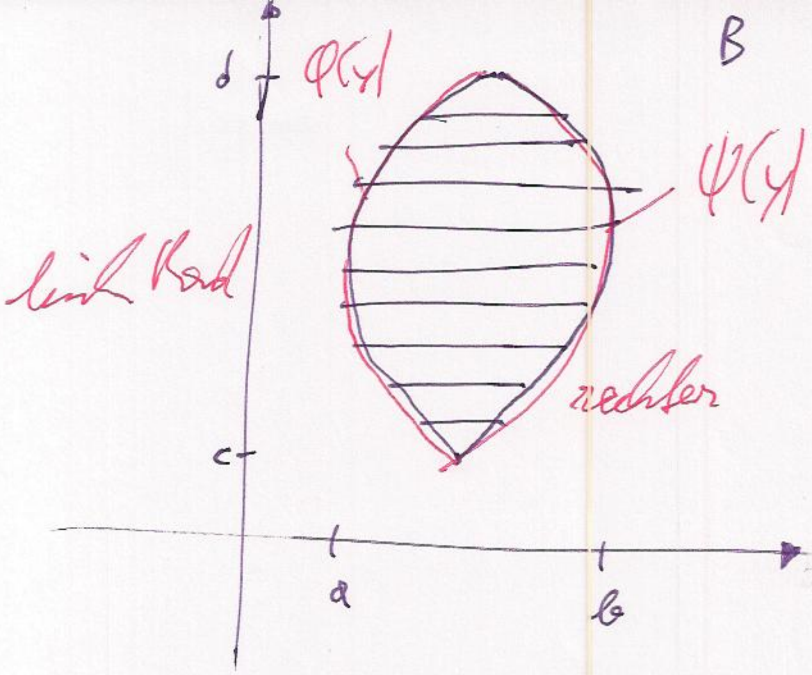
$$= y \cdot \ln y - \int 1 \cdot dy =$$

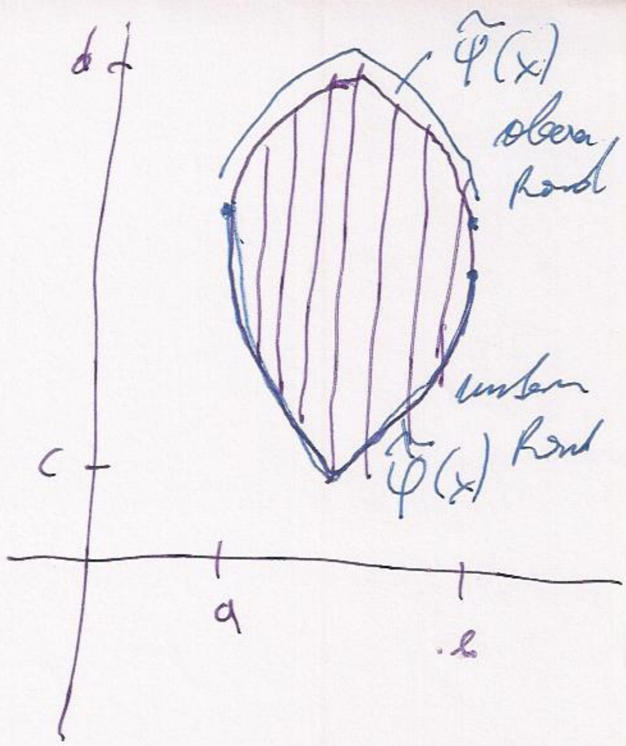
$$= y \cdot \ln y - y$$

$$(*) = \frac{(y+3)^2}{2} \cdot \ln(y+3) - \frac{(y+3)^2}{4} -$$

$$- 3 \cdot \Big((y+3) \cdot \ln(y+3) - (y+3) \Big)$$

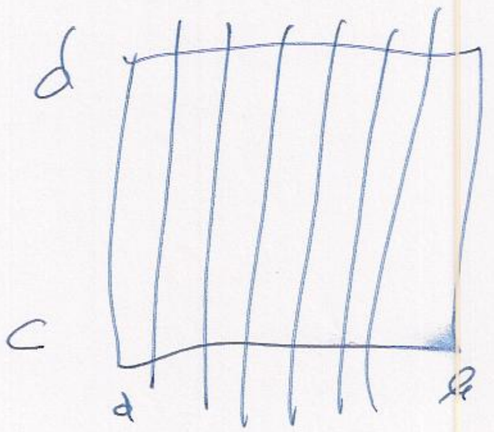
$$F(y) \Big|_2^3 = F(3) - F(2)$$

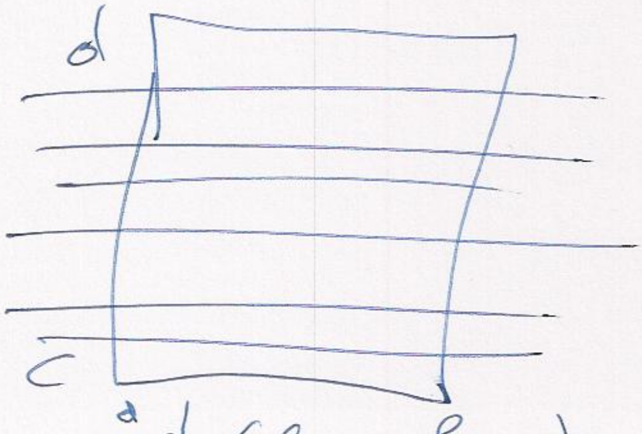




Satz: Sei $B = \{(x,y) \in \mathbb{R}^2 : \varphi(y) \le x \le \psi(y),\ c \le y \le d\}$, wobei $\varphi(y)$ und $\psi(y)$ stetige Fkt. sind. Weiters seien $\tilde\varphi(x)$ und $\tilde\psi(x)$ zwei stetige Fkt., sodass $B = \{(x,y) \in \mathbb{R}^2 : \tilde\varphi(x) \le y \le \tilde\psi(x),\ a \le x \le b\}$.

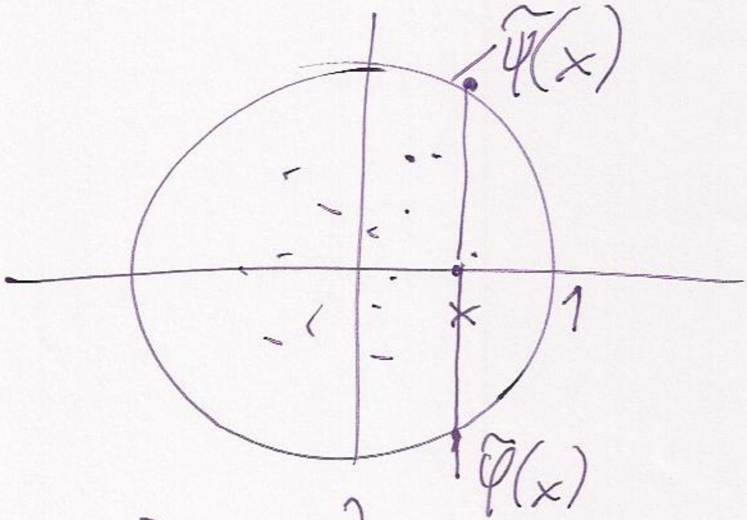
$$\Rightarrow \int_c^d \int_{\varphi(y)}^{\psi(y)} f(x,y) \cdot dx \cdot dy = \int_a^b \int_{\tilde\varphi(x)}^{\tilde\psi(x)} f(x,y) \cdot dy \cdot dx$$





$$\int_a^b \left(\int_c^d f(x,y) \cdot dy \right) dx = \int_c^d \left(\int_a^b f(x,y) \cdot dx \right) dy$$

$B =$



$$B = \{ (x,y) , \ x^2 + y^2 \leq 1 \}$$

$$\hat{\varphi}(x) = - \sqrt{1 - x^2}$$

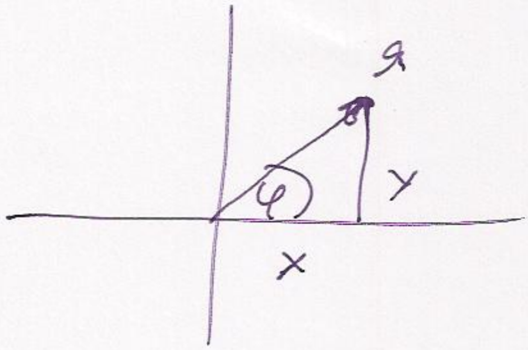
$$\tilde{\varphi}(x) = + \sqrt{1 - x^2}$$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y) \cdot dy \cdot dx$$

Polarkoor (r, φ)

$$B' = \{ (r,\varphi) : \ 0 \leq r \leq 1 , \ 0 \leq \varphi \leq 2\pi \}$$

Substituieren:



$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

---

eine Variable:

Substitutionsregel: $u = \varphi(x)$

$$\int f(u) \cdot du = \int f(\varphi(x)) \cdot \varphi'(x) \cdot dx$$

Substitutionsregel

Def.: Sei $\vec{f}(x_1, \dots, x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix}$

eine vektorwertige Fkt. auf $\mathbb{R}^n$.
Determinante der Jacobimatrix wird als
<u>Funktionaldeterminante</u> bezeichnet und
geschrieben als:

$$\det \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \dots, x_n) & \dots & \frac{\partial f_1}{\partial x_n}(x_1, \dots, x_n) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(x_1, \dots, x_n) & \dots & \frac{\partial f_n}{\partial x_n}(x_1, \dots, x_n) \end{pmatrix}$$

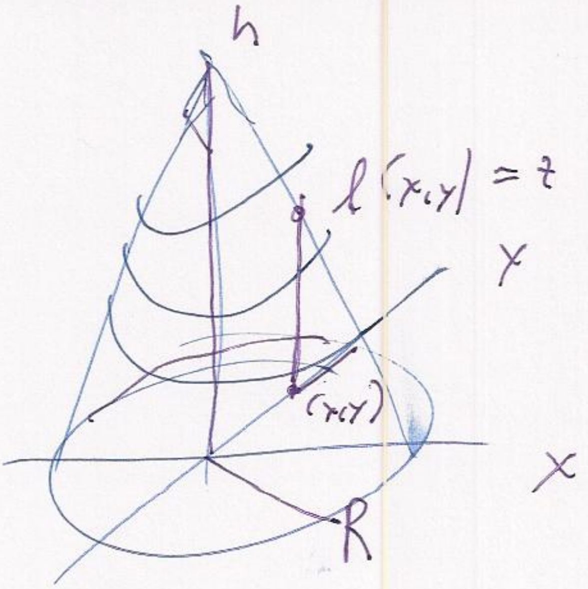
Satz: Substitutionsregel für Bereichsintegrale.

Geg. sei Bereich $B \subseteq \mathbb{R}^2$ und zwei stetig differenzierbare Fkt. $\varphi(x,y)$ und $\psi(x,y)$, die den Bereich B bijektiv auf

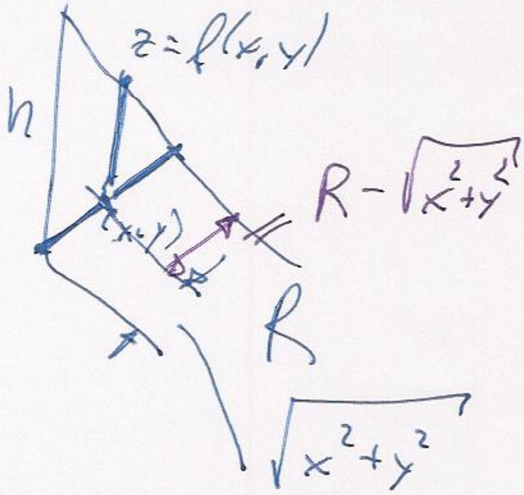
$$B' = \{(\varphi(x,y), \psi(x,y)) : (x,y) \in B\}$$

abbilden. Dann gilt:

$$\iint\limits_{B'} f(u,v)\cdot du \cdot dv = \iint\limits_{B} f(\varphi(x,y), \psi(x,y)) \cdot \overbrace{\left| \det \begin{pmatrix} \varphi_x(x,y) & \varphi_y(x,y) \\ \psi_x(x,y) & \psi_y(x,y) \end{pmatrix} \right|}^{\text{Betrag der Funktional= determ.}} \cdot dx \cdot dy$$



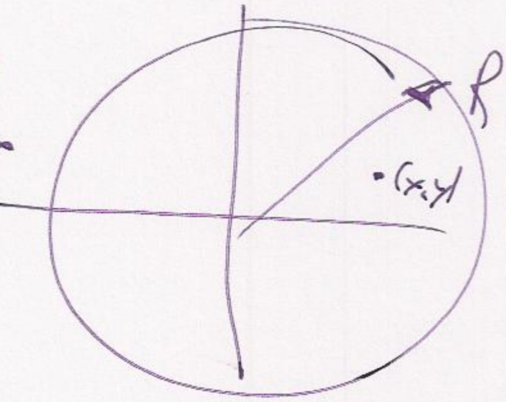
Höhe h
Radius R



$$\frac{h}{R} = \frac{z}{R - \sqrt{x^2+y^2}}$$

$$\Rightarrow \quad \overset{f(x,y)}{\overset{\shortparallel}{z}} = \frac{h}{R} \cdot \left(R - \sqrt{x^2+y^2}\right)$$

$$B = \left\{ (x,y) : x^2 + y^2 \leq R^2 \right\}$$



$$\iint\limits_B \frac{h}{R} \cdot \left(R - \sqrt{x^2+y^2}\right) \cdot dx\, dy =$$

$\Rightarrow$ Polarkoordinaten :

$$x = r \cdot \cos\varphi$$

$$y = r \cdot \sin\varphi$$

$$0 \le r \le R$$

$$0 \le \varphi \le 2\pi$$

$$= \int_{r=0}^{R} \int_{\varphi=0}^{2\pi} \frac{h}{R} \cdot \left(R - \sqrt{(r \cdot \cos\varphi)^2 + (r \cdot \sin\varphi)^2} \right) \cdot \left| \det \begin{pmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{pmatrix} \right| \cdot d\varphi \cdot dr =$$

$$= \int_0^R \int_0^{2\pi} \frac{h}{R} \cdot \left(R - \sqrt{r^2 \cdot \cos^2\varphi + r^2 \sin^2\varphi} \right) \cdot \left| \begin{matrix} \cos\varphi & -r \cdot \sin\varphi \\ \sin\varphi & r \cdot \cos\varphi \end{matrix} \right| \cdot d\varphi \cdot dr =$$

$$= \int_0^R \int_0^{2\pi} \frac{h}{R} \cdot \left(R - \sqrt{r^2 \cdot \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1}} \right) \cdot$$

$$\cdot \left| \cos\varphi \cdot r \cdot \cos\varphi - (-r \cdot \sin\varphi) \cdot \sin\varphi \right| \cdot d\varphi \cdot dr =$$

$$= \int_0^R \int_0^{2\pi} \frac{h}{R} \cdot (R - r) \cdot \left| \underbrace{r \cdot \cos^2\varphi + r \cdot \sin^2\varphi}_{r \cdot \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1}} \right| \cdot d\varphi \cdot dr$$

$$= \int_0^R \int_0^{2\pi} \frac{h}{R} \cdot (R - r) \cdot r \cdot d\varphi \cdot dr =$$

$$= \frac{h}{R} \int_0^R \int_0^{2\pi} (R \cdot r - r^2) \cdot d\varphi \cdot dr =$$

$$= \frac{h}{R} \cdot \int_0^R (R \cdot r - r^2) \cdot \varphi \Big|_0^{2\pi} \cdot dr =$$

$$= \frac{h}{R} \cdot \int_0^R (R \cdot r - r^2) \cdot 2\pi \cdot dr = *$$

$$V = \frac{2\pi \cdot h}{R} \cdot \int_0^R (R \cdot r - r^2) \cdot dr =$$

$$= \frac{2\pi \cdot h}{R} \cdot \left(R \cdot \frac{r^2}{2} - \frac{r^3}{3} \right) \Bigg|_0^R =$$

$$= \frac{2\pi \cdot h}{R} \cdot \left(R \cdot \frac{R^2}{2} - \frac{R^3}{3} \right) =$$

$$= \frac{2\pi h}{R} \cdot R^3 \cdot \underbrace{\left(\frac{1}{2} - \frac{1}{3} \right)}_{= \frac{1}{6}} =$$

$$= \frac{2\pi h \cdot R^2}{6} = \boxed{\frac{\pi \cdot h \cdot R^2}{3}}$$