(93.) z.z.: $\sum_{n\geq 0} \binom{2n}{n} x^n$ konvergent für $|x| < \frac{1}{4}$

Sei $a_n = \binom{2n}{n}$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\binom{2n}{n}}{\binom{2n+2}{n+1}} \right| = \lim_{n\to\infty} \frac{\cancel{(2n)!}}{\cancel{(2n)!}(2n+1)(2n+2)} \cdot \frac{\cancel{(n!)^2}(n+1)^2}{\cancel{(n!)^2}} =$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+5n+2} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{5}{n}+\frac{2}{n^2}} = \frac{1}{4}$$

$$\Rightarrow \sum_{n\geq 0} \binom{2n}{n} x^n \text{ konvergent für } |x| < \frac{1}{4} \quad \blacksquare$$

$$\sqrt[n]{n!}$$

$$n! \sim \frac{n^n}{e^n} \sqrt{2\pi n}$$

$$\sqrt[n]{n} \longrightarrow 1$$

$$\sqrt[n]{\frac{n^n}{e^n}\sqrt{2\pi n}} = \frac{n^{\frac{n}{n}}}{e^{\frac{n}{n}}} \cdot \underbrace{\sqrt[2n]{2\pi n}}_{\downarrow\, 1}$$

$$\sqrt[n]{n!} \sim \frac{n}{e}$$

$$\sim \frac{n}{e} \cdot 1$$

97.
$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}(x+1)^n$$

Sei $a_n = \frac{n}{n^2+1}$

I) $\lim\limits_{n\to\infty} \sqrt[n]{\left|\frac{n}{n^2+1}\right|} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{n^2+1}} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{n+\frac{1}{n}}} = 1$

II) $\lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{n}{n+1} \cdot \frac{n^2+2n+2}{n^2+1} = \lim\limits_{n\to\infty} \frac{1}{n+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} = 1$

$R = 1$

$|x+1| < 1$

· $-2 < x < 0 \Rightarrow \sum\limits_{n=1}^{\infty} \frac{n}{n^2+1}(x+1)^n$ konvergent

· $x < -2 \vee x > 0 \Rightarrow \sum\limits_{n=1}^{\infty} \frac{n}{n^2+1}(x+1)^n$ divergent

· $x = -2$:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \cdot (-1)^n$$

∘ $\lim\limits_{n\to\infty} \frac{n}{n^2+1} = \lim\limits_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0$ ✓

∘ $\frac{n+1}{n^2+2n+2} - \frac{n}{n+1} = \frac{n^3+2n+1-n^3-2n^2-2n}{\underbrace{(n^2+2n+2)(n+1)}_{>0}} < 0$ ✓

$\overset{\text{L.-K.}}{\Longrightarrow} \sum\limits_{n=1}^{\infty} \frac{n}{n^2+1} \cdot (-1)^n$ konvergent (bedingt)

· $x = 0$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{n}} \quad \text{divergent}$$

$x \in [-2; 0)$

Anmerkung:

Wurzelkriterium ist mächtiger als Quotientenkriterium.

(98.) z.z.: $\left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{b^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$, $a, b \in \mathbb{R}$

Beweis:

$$\left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{b^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} a^k \cdot \frac{1}{(n-k)!} \cdot b^{n-k}\right) =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!(n-k)!} \cdot a^k \cdot b^{n-k}\right) =$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k \cdot b^{n-k}\right) =$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k}\right) \overset{BL}{=}$$

$$\overset{BL}{=} \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} \quad \blacksquare$$

101. $a_n = \frac{2}{n}$, $b_n = \frac{1}{n^2}$, $c_n = \frac{8n^2}{4n^3+1}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{\cancel{n}} \cdot n^{\cancel{2}} = \lim_{n\to\infty} 2n = +\infty$$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{n^{\cancel{2}}} \cdot \frac{\cancel{n}}{2} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} \frac{\cancel{2}}{n} \cdot \frac{4n^3+1}{4\cancel{8}n^2} = \lim_{n\to\infty} \frac{1 + \frac{1}{4n^3}}{1} = 1$$

$$\lim_{n\to\infty} \frac{c_n}{a_n} = \lim_{n\to\infty} \frac{\overset{4}{\cancel{8}}n^2}{4n^3+1} \cdot \frac{n}{\cancel{2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{4n^3}} = 1$$

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{4n^3+1}{8n^2} = \lim_{n\to\infty} \frac{4 + \frac{1}{n^3}}{8n} = 0$$

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{8n^2}{4n^3+1} \cdot n^2 = \lim_{n\to\infty} \frac{8n}{4+\frac{1}{n^3}} = +\infty$$

$a_n = \omega(b_n) = \Omega(b_n)$

$b_n = o(c_n) = O(c_n)$

$c_n \sim a_n \Rightarrow c_n = \Theta(a_n) \Rightarrow c_n = O(a_n) \wedge a_n = O(c_n)$

$a_n \sim c_n \Rightarrow a_n = \Theta(c_n) \Rightarrow a_n = O(c_n) \wedge c_n = O(a_n)$

$b_n = o(a_n) = O(a_n)$

$c_n = \omega(b_n) = \Omega(b_n)$

$a_n = o(d_n) \Rightarrow a_n = O(d_n)$

$\frac{a_n}{d_n} \to 0 \Rightarrow \left|\frac{a_n}{d_n}\right| \leq C \Rightarrow |a_n| \leq C \cdot |d_n|$

$\sin n = O(1)$

aber: $\sin n \neq o(1)$

106. Stirling'sche Approximationsformel:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

Z.Z.: $\binom{3n}{n} \sim \left(\frac{27}{4}\right)^n \sqrt{\frac{3}{4\pi n}}$

Beweis:

$$\binom{3n}{n} = \frac{(3n)!}{n!(2n)!} \sim \frac{(3n)^{3n} \cdot \cancel{e^{-3n}} \sqrt{\cancel{2\pi} 3\cancel{n}}}{n^n \cancel{e^{-n}} \sqrt{\cancel{2\pi n}} \cdot (2n)^{2n} \cdot \cancel{e^{-2n}} \sqrt{2\pi 2n}} =$$

$$= \frac{(3n)^{3n}}{n^n (2n)^{2n}} \cdot \sqrt{\frac{3}{4\pi n}} =$$

$$= \left(\frac{(3n)^3}{n \cdot (2n)^2}\right)^n \cdot \sqrt{\frac{3}{4\pi n}} =$$

$$= \left(\frac{27 n^{\cancel{3}}}{\cancel{n} \cdot 4 n^{\cancel{2}}}\right)^n \sqrt{\frac{3}{4\pi n}} =$$

$$= \left(\frac{27}{4}\right)^n \cdot \sqrt{\frac{3}{4\pi n}} \quad \blacksquare$$

$$\frac{a_n \sim b_n \qquad c_n \sim d_n}{\Rightarrow a_n \cdot c_n \sim b_n \cdot d_n}$$

$$\frac{a_n}{c_n} \sim \frac{b_n}{d_n}$$

109. Eulersche Formel:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

z.z.: $\cos(u+v) = \cos u \cos v - \sin u \sin v$

Beweis:

$$\cos(u+v) = \mathrm{Re}\left(e^{i(u+v)}\right) = \mathrm{Re}\left(e^{iu} \cdot e^{iv}\right) =$$

$$= \mathrm{Re}\left((\cos u + i\sin u)\cdot(\cos v + i\sin v)\right) =$$

$$= \mathrm{Re}\left(\cos u \cos v \overset{+i^2}{-} \sin u \sin v + i(\sin u \cos v + \cos u \sin v)\right) =$$

$$= \cos u \cos v - \sin u \sin v \quad \blacksquare$$

(113.) Bekannt: Rechenregeln für die Exponentialfunktion e^x

z.z.: $\cdot \ln(xy) = \ln x + \ln y$

$\cdot \ln(x^y) = y \ln x$

Beweis:

$\cdot\; e^{\ln(xy)} = xy = e^{\ln x} \cdot e^{\ln y} = e^{\ln x + \ln y} \xRightarrow{(\exp(x) \text{ bijektiv})} \ln(xy) = \ln x + \ln y \;\blacksquare$

$\cdot\; e^{\ln(x^y)} = x^y = (e^{\ln x})^y = e^{y \ln x} \xRightarrow{(\exp(x) \text{ bijektiv})} \ln(x^y) = y \ln x \;\blacksquare$