VU Discrete Mathematics

Exercises for 4th November 2025

19) Let A, B be two finite sets with $|A| = n$ and $|B| = k$. How many injective mappings $f : A \to B$ are there? Furthermore, show that the number of surjective mappings $f : A \to B$ equals $k!S_{n,k}$.

20) The n-th Bell number equals the number of set partitions of $\{1, 2, \ldots, n\}$. We set $B_0 := 1$. Prove the following identities:

$$B_n = \sum_{k=0}^{n} S_{n,k} \qquad \text{and} \qquad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

21) Solve the following recurrence using generating functions:

$$a_{n+1} = a_n + (n+1)^2 \text{ for } n \geq 0,\ a_0 = 1.$$

22) Consider the following context-free grammar: $S \to aSbS|\varepsilon$. This defines a formal language $\mathcal{L}$ which consists of all words w over the alphabet $\Sigma = \{a, b\}$ such that either (a) w starts with a followed by a word from $\mathcal{L}$, then a b follows, which is itself followed by another word of $\mathcal{L}$, or (b) w is the empty word. Compute the number of words in $\mathcal{L}$ that consist of n letters. Do this by finding a combinatorial structure that specifies $\mathcal{L}$ and analyzing the generating function of that structure.

23) Consider a regular $(n+2)$-gon A, say, with the vertices $0, 1 \ldots, n+1$. A triangulation is a decomposition of A into n triangles such that the 3 vertices of each triangle are vertices of A as well. Show that the set $\mathcal{T}$ of triangulations of regular polygons can be described as a combinatorial construction satisfying

$$\mathcal{T} = \{\varepsilon\} \cup \mathcal{T} \times \Delta \times \mathcal{T}$$

where Δ is the set containing the unique triangulation of a single triangle and ε denotes the empty triangulation (consisting of no triangle and corresponding to the case $n = 2$). What is the number of triangulations of A?

24) Determine all solutions of the recurrence relation:

$$a_n - 2na_{n-1} + n(n-1)a_{n-2} = 2n \cdot n!, \text{ for } n \geq 2,\ a_0 = a_1 = 1.$$

Hint: Use exponential generating functions.