2. ÜBUNG
104.272 Discrete Mathematics mit Lösungen

(11) A graph $H = (V', E')$ is an induced subgraph of $G = (V, E)$, if $V' \subseteq V$ and any edge in G connecting two vertices a,b in V' is in E'.

Let G be a connected simple graph that does not have path or cycle with four vertices as an induced subgraph. Show that G has a vertex adjacent to all other vertices. (Hint: consider a vertex of maximum degree.)

Lösung: *Let u be a vertex of maximum degree and suppose that (u, w) is not in G. Let P be a shortest path from u to w. P cannot have more than three vertices, because then its vertices would induce a path on four vertices. (Suppose $P = u, v_1, v_2, w$, then (v_1, w) and (u, v_2) cannot be edges because P is a shortest path.)*

Suppose the shortest path is $P = u, v, w$. We show that the degree of v is larger than the degree of u, contradicting our hypothesis. Suppose that $t \neq v$ is adjacent to u. Since the four vertices t, u, v, w should not induce a path or cycle, and $(u, w) \notin G$, we have that $(t, v) \in G$. Thus every vertex adjacent to u is also adjacent to v, but additionally v is adjacent to w.

(12) Let T be a tree without vertices of degree 2. Show that T has more leaves than internal nodes using the handshaking lemma.

Lösung: *See Exercise 14.*

(13) Let G be a connected graph with an even number of vertices. Show that G has a spanning (but not necessarily connected) subgraph with all vertices of odd degree. Show that this is not necessarily the case for arbitrary graphs.

Lösung: *Select a tree. If there is a vertex of even degree, consider the (even number of) subtrees obtained by deleting this vertex. The total number of vertices in these subtrees is odd, so there must be an odd number of trees with an even number of vertices. To obtain the subgraph from G, delete the edges leading to the even subtrees.*

(14) Let T be a tree and let n_d be the number of vertices of degree d in T. Show that the number of leaves of T equals

$$2 + \sum_{d \geq 3} (d-2)n_d.$$

Lösung: *Adding $(d-2)n_1$ on both sides this is equivalent to $0 = 2 + \sum_d (d-2)n_d$ which in turn is equivalent to $\sum_d dn_d = 2|E|$.*

(15) Show that the number of spanning trees of the complete graph on n vertices K_n is n^{n-2}, using the matrix tree theorem. Hint: To compute the determinant of the resulting matrix, add all rows except the first one to the first row. Then add the first row of this new matrix to the other rows.

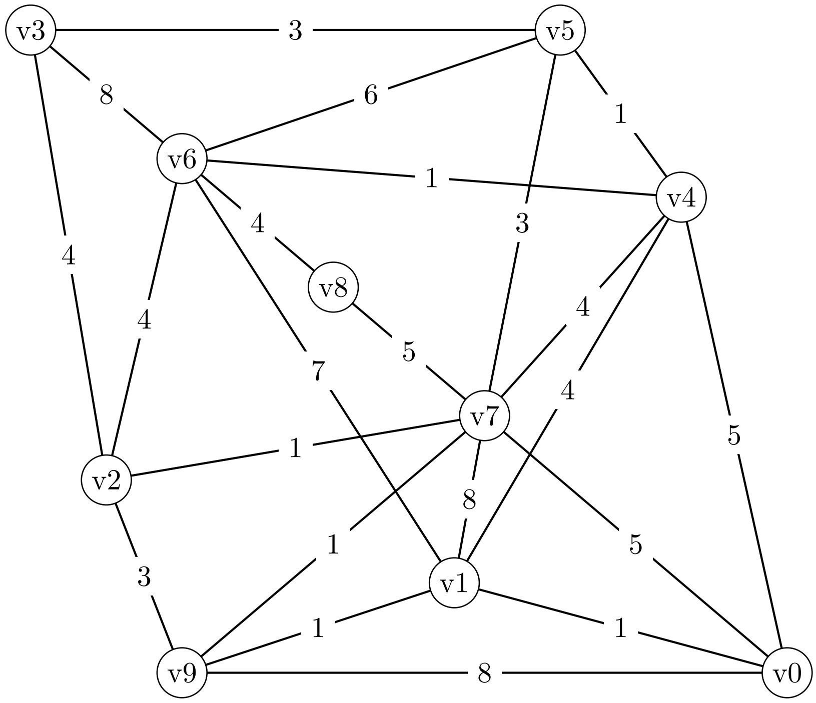
(16) Let G be a connected graph with n vertices. Let G_T be the graph having the spanning trees of G as vertices, with two vertices s and t being adjacent if and only if the corresponding spanning trees in G share precisely $n-2$ edges.

Show that G_T is connected. Hint: look at the proof of the correctness of Kruskal's algorithm given in the lecture.

Lösung: *Let R and T be any pair of spanning trees with $|R \cap T| < n-2$. We show that there is a tree S with $|R \cap S| = n-2$ and $|R \cap T| < |S \cap T|$.*

Let $E_1 = R \cap T$, $E_2 = T \setminus R$ *and* $E_3 = R \setminus T$. *Let* e *be any edge from* E_2. *Use Kruskal's algorithm to construct* S *by completing* $E_1 \cup \{e\}$ *(which is acyclic) to a spanning tree using only elements of* E_3. *Then* $|S| = n - 1$, *because* S *is a tree, and therefore* $|R \cap S| = n - 2$ *and* $|R \cap T| < |S \cap T|$.

(17) Compute a minimum and a maximum spanning tree of the graph below using Kruskal's algorithm.



(18) Find an Eulerian circuit in the directed graph below as follows:
   (a) for each vertex u, specify an ordering of the arcs that leave u, such that the arc leaving u in the spanning tree (indicated with the thick arcs) comes last.
   (b) beginning at the vertex 6, construct an Eulerian circuit by always exiting the current vertex u along the next unused arc in the ordering specified.