Solutions to Old Exams

January 2024

1 Exam 13.09.2022

Task 1

Formalize the following sentence: "If every politician is a showman and no showman is sincere, then there exists a politician that is insincere." If the sentence is true then prove it, otherwise state a counterexample.

Solution. *We use the following symbols and intended meaning:*

$$\begin{aligned} &P(x) \dots x \text{ is a politician}, \\ &S(x) \dots x \text{ is a showman}, \\ &\mathcal{S}(x) \dots x \text{ is sincere}. \end{aligned}$$

Note that one could express the formula more compactly by having a symbol for insincerity directly.

The formula is then best described by

$$(\forall x(P(x) \to S(x)) \land \forall x(S(x) \to \neg\mathcal{S}(x))) \to \exists x(P(x) \land S(x)).$$

This sentence is not true, in the sense that it is not valid. For instance, it is not fulfilled by the structure $\mathcal{A} = (D_{\mathcal{A}}, I_{\mathcal{A}})$, where $D_{\mathcal{A}} = \{0\}, P^{\mathcal{A}} = S^{\mathcal{A}} = \mathcal{S}^{\mathcal{A}} = \emptyset$ (together with an arbitrary environment).

Task 2

Consider the following formula $\exists x_1 \exists x_2 (B(x_1, x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2)))$. If the formula is valid present a proof using sequent calculus, otherwise state a counterexample.

Solution. *It is easy to see that this formula is valid. Hence, we provide a proof using sequent calculus. To keep the proof short and sweet, we set $\varphi \coloneqq \exists x_1 \exists x_2 (B(x_1, x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2)))$. The proof then is:*

$$
\begin{array}{cl}
B(f(y_1), y_2) \vdash (B(f(y_1), y_2)), \forall y_1 \forall y_2 (B(f(y_1), y_2)) & \\ \hline
\vdash (B(f(y_1), y_2)), B(f(y_1), y_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2)) & \to R \\ \hline
\vdash (B(f(y_1), y_2)), \exists x_2 (B(f(y_1), x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2))) & \exists R \\ \hline
\vdash (B(f(y_1), y_2)), \exists x_1 \exists x_2 (B(x_1, x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2))) & \exists R \\ \hline
B(x_1, x_2) \vdash B(f(y_1), y_2), \varphi & WL \\ \hline
B(x_1, x_2) \vdash \forall y_2 (B(f(y_1), y_2)), \varphi & \forall R \\ \hline
\vdash B(x_1, x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2)), \varphi & \forall R \\ \hline
\vdash \exists x_2 (B(x_1, x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2))), \varphi & \exists R \\ \hline
\vdash \exists x_1 \exists x_2 (B(x_1, x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2))), \varphi & \exists R \\ \hline
\vdash \exists x_1 \exists x_2 (B(x_1, x_2) \to \forall y_1 \forall y_2 (B(f(y_1), y_2))) & CR
\end{array}
$$

Observe that the WL step could be omitted, and was only included to improve the exposition.

Task 3 + 4

Consider the following two sets

$$I_1 = \{j \mid \Phi_j(j) \downarrow \text{ within } k \text{ steps}\} \quad \text{and } I_2 = \{j \mid \Phi_j(j) \downarrow \text{ with } j \in k\}.$$

Are the two sets recursive or r.e.? Can we use Rice Theorem? Motivate your answer and consider the two cases where

(a) $k = 1, \ldots, 4$,

(b) $k = \mathcal{N}$.

Solution. *We assume that $I_1 = \{j \mid \Phi_j(j) \downarrow$ within k' steps where $k' \in k\}$. We first consider the case where $k = 1, \ldots, 4$. In this case I_1 is a recursive set. Given any natural number, we can check whether $\Phi_j(j)$ terminates within at most four steps, if terminates within these steps, we output 1, otherwise, we output 0. We clearly only need to run $\Phi_j(j)$ for at most four steps, hence the computation terminates always.*

Regarding Rice theorem, it is clear that the set I_1 is non-trivial. However, terminating within at most 4 steps is not an extensional property, and hence, it cannot be applied. It is clear that, for instance for the constant zero function, we can provide an algorithm that terminates within 4 steps, and an algorithm that does not, e.g. by doing 5 unnecessary computations before outputting the number 0.

We proceed by considering the set I_2. It is clear that $I_2 \subseteq [1 \ldots 4]$ in this case. Hence, the set is recursive, as it is one of 2^4 sets, all of which are trivially recursive.

Next, we consider the case where $k = \mathcal{N}$. In this case, the set I_1 is not recursive, but we cannot apply Rice theorem. The reason is, that the set is not extensional: Indeed, the set makes use of the indices of the function in its own definition, which does generally not lead to extensional sets. We can prove that it is not recursive. We know that there exists a function that only terminates on its own index. Hence, this function is in the set. However, equivalent functions on different indices are not in the set.

So, we must show that the set is not recursive differently. Assume that the set is recursive. Then, the following function is computable:

$$f(x) = \begin{cases} 1 & \textit{if } \Phi_x(x) \downarrow, \\ 0 & \textit{otherwise.} \end{cases}$$

However, this function is closely related to the halting problem, and we know that it is not computable.

On the other hand, the set is r.e. To show this, we proof that it is semi-decidable. That is, we can provide an algorithm that always outputs 1 if the input n is in I_1, and does not terminate otherwise. The algorithm that simulated the execution of $\Phi_j(j)$, outputs 1 if the execution terminates is exactly this algorithm.

Finally, consider set I_2. Clearly, when $k = \mathcal{N}$, we have $I_1 = I_2$, and hence we already elaborated on this case.

Task 5

Let G be the modal formula $\Diamond\Diamond\neg A \rightarrow \neg\Box A$. Prove or refute:

1. G is valid in every transitive frame,
2. If $F \models G$ for a frame F, then F is transitive.

Solution. *We first proof that G is valid in every transitive frame. Consider an arbitrary model of an arbitrary transitive frame. Furthermore, let w be an arbitrary world in which $\Diamond\Diamond\neg A$ holds. Then, w can see a world w', such that $\Diamond\neg A$ holds in w'. Ergo, w' can see a world w'' such that $\neg A$ holds in w''. Because the frame is transitive, w can also see w''. Then, w can see a world in which $\neg A$ is not true, which immediately gives that $\neg\Box A$ is true in w.*

Now, we proof that if $F \models G$, then F is transitive. In a proof by contradiction, let F be any non-transitive frame. Because F is non-transitive, there are worlds w, w', w'' in F such that w can see w', w' can see w'', but w cannot *see w''. We provide an interpretation such that under this interpretation, G does not hold in w. Indeed, consider the interpretation that sets p true in all worlds except for w''. Then, $\Diamond\Diamond\neg p$ holds in w, because w can see w', w' can see w'', and $\neg p$ holds in w''. However, p holds in every world except for w''. In particular, this means that p holds in every world that w can see, and hence $\Box p$ holds in w, which means that $\neg\Box A$ does not.*

Task 6

Compute all Robinson resolvents of two clauses.

Solution. *Task description is not complete.*

Task 7

Use the proof of the ADRF theorem to show that the function

$$f(g(x) \cdot g(y), x + y) + 2$$

is arithmetically definable, if f and g are arithmetically definable.
Hint: Specify the witnessing formula, following slide 23 of the last set of slides (on incompleteness).

Solution. *We specify the witnessing formula. Let F be the formula of function f, G the formula specifying g.*[1]

$$\exists v_{gx} \exists v_{gy} \exists v_. \exists v_+ \exists v_f (G(v_1, v_{gx}) \wedge G(v_2, v_{gy}) \wedge \cdot(v_{gy}, v_{gx}) = v_. \wedge +(v_1, v_2) = v_+ \wedge F(v_., v_+, v_{v_f}) \\ v_3 = +(ss0, v_f))$$

[1] In the exam it might be wiser to use v_x for some natural number x instead of other symbols, as these are technically not in the language.

2 Exam 24.06.2021

Task 1

Consider the following reasoning:

(a) Every rigorous person admires some mathematician.

(b) Every mathematician admires some rigorous person.

(c) Alf admires only himself.

(d) Either Alf is not a mathematician or Alf is a rigorous person.

Solution. *We use the following symbols and describe the intended meaning:*

$$\begin{aligned}&R(x)\ldots x \textit{ is rigorous},\\ &M(x)\ldots x \textit{ is a mathematician},\\ &A(x,y)\ldots x \textit{ admires } y,\\ &a\ldots \textit{ Alf.}\end{aligned}$$

We do not use a predicate to express that someone is a person, since all statements are about people anyways. Then, the statements correspond to the following formulas:

(a) $\forall x(R(x) \rightarrow \exists y(M(y) \wedge A(x,y)))$,

(b) $\forall x(M(x) \rightarrow \exists y(R(y) \wedge A(x,y)))$,

(c) $A(a,a) \wedge \forall x(A(a,x) \rightarrow x = a)$,

(d) $\neg M(a) \vee R(a)$.

Even though statement (d) includes the word either, we would argue that the standard logical or best described the intended meaning of the sentence. It is easy to see that (d) is indeed a consequence of (a)-(c). We prove the claim using semantics. Let $\mathcal{A}$ *be an arbitrary structure such that* $\mathcal{A} \models (a), \mathcal{A} \models (b), \mathcal{A} \models (c)$. *Observe that the environment does not matter because the formulas are all closed.*

Case 1: $\mathcal{A} \models \neg M(a)$. *Then trivially* $\mathcal{A} \models \neg M(a) \vee R(a)$.

Case 2: $\mathcal{A} \not\models \neg M(a)$. *Hence,* $\mathcal{A} \models M(a)$. *Consider formula (b), we have* $\mathcal{A} \models (b)$. *Furthermore, we have just established that* $\mathcal{A} \models M(a)$. *Hence, if we assign variable* x *to* $a^{\mathcal{A}}$, *the resulting structure-environment pair must evaluate the formula* $\exists y(R(y) \wedge A(x,y))$ *to true. So, there is an element* $d \in D_{\mathcal{A}}$, *such that* $R^{\mathcal{A}}(d)$ *and* $A^{\mathcal{A}}(a,d)$. *However, by (c), we obtain that* $d = a$. *Hence, we have* $R^{\mathcal{A}}(d) \Leftrightarrow R^{\mathcal{A}}(a)$, *and it follows that* $\mathcal{A} \models R(a)$.

Note that a semantic proof was the obvious choice here, as our proof systems do not support equality, and we would need additional formulas to compile the predicate properly.

Task 2

Prove or disprove the following statements:

- $\models \exists x(\neg A(x) \vee \forall x(A(x)))$,
- $(\exists x A(x) \rightarrow \exists x B(x)) \models \forall x(A(x) \rightarrow B(x))$.

Solution. *It is easy to see that the first statement holds. We provide a proof using the sequent calculus. Since both LK, LK' are sound, we can clearly use rules, axioms from either calculus.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \vdash A(a)}{\vdash A(a), \neg A(a)}}{\vdash A(a), \neg A(a) \lor \forall x(A(x))}}{\vdash A(a), \exists x(\neg A(x) \lor \forall x(A(x)))}}{\vdash \forall x(A(x)), \exists x(\neg A(x) \lor \forall x(A(x)))}}{\vdash \neg A(a) \lor \forall x(A(x)), \exists x(\neg A(x) \lor \forall x(A(x)))}}{\vdash \exists x(\neg A(x) \lor \forall x(A(x)))}$$

If the second statement was true, the harmony of nature would be disturbed. Hence, it must be false. Consider the structure $\mathcal{A} = (D_\mathcal{A}, I_A)$, where $D_\mathcal{A} = \{0, 1\}$ and $A^\mathcal{A} = \{1\}, B^\mathcal{A} = \{0\}$. It is easy to see that $\mathcal{A} \models (\exists x A(x) \to \exists x B(x))$, because there exists a domain element for which $A^\mathcal{A}$ is false (namely, element 0). However, we also have $\mathcal{A} \not\models \forall x(A(x) \to B(x))$: For element 1 we have that $A^\mathcal{A}$ holds, but $B^\mathcal{A}$ does not.

Task 3

Are the sets

- $\{x \mid \exists y \Phi_x(y) \neq 5\}$,
- $\{x \mid \phi_x(0) \downarrow \land x \leq 5\}$,

recursive, r.e. or none of them? Motivate your answers.

Solution. *This set is not recursive by Rice theorem. Indeed, the set collects the indices of all functions, such that there exists an input for which the output is not 5. In other words, this set collects the indices of all functions that are not the constant 5 function. It is clear that the set is extensional, because this is solely a property of the considered function. Furthermore, some functions are in the set, and some are not, so, the set is not recursive.*

If we consider $\phi_x(y) \downarrow$ to fulfill $\phi_x(y) \neq 5$, then the set is not r.e. However, the proof is difficult because the complement is also not r.e, so we cannot apply Post theorem. We use a different interpretation instead.

The set is r.e., assuming that the set is meant as

$$\{x \mid \exists y \Phi_x(y) \neq 5 \land \Phi_x(y) \downarrow\}.$$

This assumption is reasonable as well, given that $\Phi_x(y) \downarrow$ is not even a well-defined output. Then, we can show that the set is semi-decidable, and therefore also r.e. Consider the following algorithm:

It is easy to see that this algorithm is correct. Clearly, when there is an input y such that $\Phi_x(y) \downarrow \land \Phi_x(y) \neq 5$, there is a finite bound on the execution time on $\Phi_x(y)$. By employing dove-tailing, we will eventually try this input for long enough, and output 1. On the other hand, if there is no such y, the algorithm clearly never terminates.

Now, we move our attention to the second set. Clearly, this set is a subset of $[0 \ldots 5]$. Hence, it is one of at most 2^6 recursive sets, and therefore also recursive.

Algorithm 1: Semi-decidability Algorithm

Input: integer x

$k := 0$
while *True* **do**
    $(i, t) := (\alpha_1(k), \alpha_2(k))$
    Execute $\phi_x(i)$ for at most t steps
    if $\Phi_x(i)$ *terminated and had an output different from* 5 **then**
        return *1*
    end
end

Task 4

Consider the following statements:

1. If I is infinite, then I is extensional.

2. I is extensional then I is infinite.

Are they true? (Motivate your answer)

Solution. *The first statement is clearly false. Indeed, there exist sets that are not extensional, but infinite. An example would be* $\{x \mid \Phi_x(0)$ *terminates within* 5 *steps*$\}$*. This set is infinite, because there e.g. exist infinitely many Turing machines that output* 0 *within one step. These Turing machines can still be syntactically different, for example, by including states that cannot be reached. However, there are also infinitely many Turing machines that output* 0 *within more than* 5 *steps, by doing unnecessary computations and then outputting* 0*. Hence, the set is not extensional, but infinite.*

Also, the set $\{i \mid i$ *is odd*$\}$ *is not extensional. By the way the enumerations are defined, it is not the case that equivalent functions always have the same parity.*

The second statement is clearly true, and a direct consequence of the padding Lemma.

Task 5

Compute all factors of the clause $p(f(y), f(x)) \vee p(f(x), z) \vee p(y, z)$.

Solution. *Recall that a factor is a p-reduct of a G-instance of a clause, and furthermore, C is a factor of itself. Hence, the first factor we have is the clause itself,* $p(f(y), f(x)) \vee p(f(x), z) \vee p(y, z)$.

Next, we compute the G-instances of the clause and the corresponding p-reducts. For this purpose, we consider any non-empty subset of the literals of the clause, and check if we can unify them.

- $|\mathcal{A}| = 1$*: This gives us the clause itself as the G-instance, however, the clause has no p-reduct.*

- $\mathcal{A} = \{p(f(y), f(x)), p(f(x), z)\}$*: We can unify these using the mgu* $\sigma = \{y \to x, z \to f(x)\}$*. The resulting factor is then* $p(f(x), f(x)) \vee p(x, f(x))$*.*

- $\mathcal{A} = \{p(f(y), f(x)), p(y, z)\}$*: Cannot be unified (occurs check).*

- $\mathcal{A} = \{p(f(x), z), p(y, z)\}$: *Can be unified using mgu* $\sigma = \{y \to f(x)\}$. *The resulting factor is then* $p(f(f(x)), f(x)) \lor p(f(x), z)$.
- $|\mathcal{A}| = 3$. *Clearly, we cannot unify all literals of the clause, because we could already not unify a subset.*

Task 6

Let G be the modal formula $\Diamond\neg A \lor \Diamond\Box A$. Prove or refute:

1. G is valid in every reflexive frame.
2. If $\mathcal{F} \models G$ for a frame $\mathcal{F}$ then $\mathcal{F}$ is reflexive.

Solution. *Even a donkey could see that G is valid in every reflexive frame. Consider an arbitrary reflexive frame* $\mathcal{F}$*, and an arbitrary interpretation of said frame. Furthermore, let* w *be a world of the frame. Then,* $\Diamond\neg A$ *might hold. Otherwise,* w *does not see a world in which* $\neg A$ *holds. Hence, A holds in every world seen by* w*, and this tells us that* $\Box A$ *holds in* w*. Since* w *can see itself, this implies that also* $\Diamond\Box A$ *holds in* w*.*

However, there are non-reflexive frames in which the formula is valid. The next figures illustrate exactly such a frame.

We briefly argue why the formula is valid in the frame. First, consider w_2*. If A is true in* w_2*, then* w_2 *can see only worlds in which A holds, and hence,* $\Box A$ *is true in* w_2*. Then* $\Diamond\Box A$ *also holds in* w_2 *because it can see itself.*

If A is false in w_2*, then* $\Diamond\neg A$ *holds in* w_2 *for the same reason.*

Now, consider w_1*. If* $\neg A$ *holds in* w_2*, then clearly* $\Diamond\neg A$ *holds in* w_1*.*

If A holds in w_2*, then, we know that* $\Box A$ *holds in* w_2*, and hence,* $\Diamond\Box A$ *holds in* w_1*.*

Task 7

We provide the witness formula, using that F is the formula for f, and G the formula for g.

$$\exists v_2, v_f, v_{gf}, v_g, v_{f'}, v_+ (G(v_1, v_g) \land F(v_1, v_f) \land G(v_f, v_{gf}) \land v_+ = +(v_g, v_{gf})$$
$$F(v_+, v_{f'}) \land v_2 = +(v_{f'}, sss0))$$

3 Exam 04.05.2021

Task 1

Consider the following reasoning (involving 'megabarbers', 'bureaucratic barbers' - 'burobarber' in short - and 'megaburobarbers' who are both megabarbers and burobarbers):

(a) From 'every megabarber shaves all those who do not shave themselves' follows 'there does not exist any megabarber'

(b) From 'every burobarber shaves only those who do not shave themselves' follows 'there does not exist any burobarber'.

(c) From 'every megaburobarber shaves all those and only those who do not shave themselves' follows 'there does not exist any megaburobarber'.

Formalize (a), (b) and (c), and establish which of these consequence claims are correct. (In each case, provide a proof or a countermodel.)

Solution. *We use the same symbols with the following intended meaning for all formalizations:*

$$
\begin{aligned}
&M(x) \ldots x \textit{ is a megabarber},\\
&B(x) \ldots x \textit{ is a burobarber},\\
&MB(x) \ldots x \textit{ is a megaburobarber},\\
&S(x,y) \ldots x \textit{ shaves } y.
\end{aligned}
$$

We start by formalizing (a):

$$\forall x(M(x) \to \forall y(\neg S(y,y) \to S(x,y))) \models \neg\exists x M(x)$$

(b):

$$\forall x(B(x) \to \forall y(S(x,y) \to \neg S(y,y))) \models \neg\exists x B(x)$$

and (c):

$$\forall x(MB(x) \to \forall y((S(x,y) \to \neg S(y,y)) \wedge (\neg S(y,y) \to S(x,y)))) \models \neg\exists x MB(x).$$

Regarding the correctness of (a), The consequence claim is not correct. Consider the structure $\mathcal{A} = (D_\mathcal{A}, I_\mathcal{A})$ with $D_\mathcal{A} = \{1\}, S^\mathcal{A} = \{(1,1)\}, M^\mathcal{A} = \{1\}$. Formally, one could evaluate the truth value of the function under this interpretation (with an arbitrary environment), but we present an intuitive argument here. Indeed, there is a domain element for which $M^\mathcal{A}$ holds, and hence it is clear that the right-hand side is not fulfilled by the interpretation. On the other hand, the left-hand side is fulfilled: Clearly, $M^\mathcal{A}$ holds for the only domain element 1, however, $\neg S(1,1)$ does not. Hence, the inner implication on the left-hand side evaluates to true, and so does the whole formula.

Now, we focus on the correctness of (b). Once again, the claim is not correct. Consider the structure $\mathcal{A} = (D_\mathcal{A}, I_\mathcal{A})$ with $D_\mathcal{A} = \{1\}, S^\mathcal{A} = \emptyset, B^\mathcal{A} = \{1\}$. It is easy to see that the right-hand side is false under this structure. However, the left-hand side is true: There is only one domain element, and while $B^\mathcal{A}$ is true for it, $S^\mathcal{A}(1,1)$ is false for it. Hence, the inner implication evaluates to true, and so does the formula.

On the other hand, the claim of (c) is indeed correct. We provide a semantic argument why this is true. Consider an arbitrary structure $\mathcal{A} = (D_\mathcal{A}, I_\mathcal{A})$ such that $\mathcal{A} \models \exists x MB(x)$. Hence, there is an element $d \in D_\mathcal{A}$ such that $MB^\mathcal{A}(d)$ is the case. Now, we take a look at the left-hand side of the consequence claim, and assume that the left-hand side is true under the structure. The formula is all-quantified, and hence, the formula we obtain by mapping x to d must be true. Clearly, $MB(x)$ evaluates to true under the considered interpretation and environment. Hence, the consequence of the implication must also be true. However, this cannot be the case. Indeed, the consequence must be true when both x and y are mapped to d. The resulting statement is clearly contradictory: If $S^\mathcal{A}(d,d)$ holds, then $S^\mathcal{A}(d,d)$ cannot hold, and the other way around. Hence, our assumption that the left-hand side is fulfilled must be false, proving that the consequence claim is true.

Task 2

Prove or disprove the following statement $\models \exists x(\exists y A(y) \to A(x))$.

Solution. *Even an unicellar could see that the formula is valid. We provide a proof in the sequent calculus.*

$$\dfrac{\dfrac{\dfrac{A(x) \vdash A(x)}{\vdash A(x) \to A(x)} \to R}{\vdash \exists y A(y) \to A(x)} \exists R}{\vdash \exists x(\exists y A(y) \to A(x))} \exists R$$

Task 3

Recall that a function f is 1-1 if $f(x) = f(y) \Rightarrow x = y$. Is the set

$$\{x \mid \Phi_x \text{ is 1-1}\}$$

recursive, r.e. or none of them? (Motivate your answer.)

Solution. *The set is clearly extensional, since we consider a property of the function only, regardless of the used representation. Furthermore, some functions are 1-1 and some are not. Hence, the set is not recursive by Rice theorem.*

Next, we show that the complement of the set is r.e., proving that the set is not using Post theorem. Indeed, the complement contains the indices of all functions that are not 1-1, that is, there exist two different inputs with the same output. We will assume that non-termination does not count as an output in this case. Then, we can give the following algorithm to compute whether an input x is in the complement or not:

Algorithm 2: Algorithm proving that the complement of the set is r.e.

```
Input: integer x
k := 0
S := ∅
while True do
    (i, t) ← (α1(k), α2(k))
    Let Φx(i) run for at most t steps
    if The computation terminated with output o then
        S := S ∪ {(i, o)}
        if There is a pair (i', o), i' ≠ i in S then
            return 1
        end
    end
end
```

Using the technique of dove-tailing, we simply let any input run for any finite amount of time. If two different inputs lead to the same output, we will hence detect this. On the other hand, if the function is 1-1, the algorithm never terminates.

Task 4

Let g be a total computable function. Is the set

$$H = \{x \mid \neg\exists y \text{ s.t. } \Phi_{g(y)} = x\}$$

recursive, r.e. or none of them?

Solution. *The task description is not correct, and it is not clear what is meant.*

Can we solve the problem using some reasonable assumption about what is actually meant?

Task 5

Show by Robinson-resolution that the clause set $\{C, D, E\}$ is unsatisfiable, where

$$\begin{aligned} C &= q(f(x), c) \vee q(y, c), \\ D &= q(f(y), f(x)) \vee \neg q(f(x), y), \\ E &= \neg q(f(f(x)), f(c)). \end{aligned}$$

Specify all used factors, MGUs, and unified literals. (c is a constant, x, y are variables).

Solution. *We state a Robinson-refutation, slightly misusing the concept, we begin with variable disjoint variants of the clauses:*

1	$q(f(x_1), c) \vee q(y_1, c)$	*variant of* C
2	$q(f(y_2), f(x_2)) \vee \neg q(f(x_2), y_2)$	*variant of* D
2.5	$q(f(x_1), c)$	*factor of 1 using mgu* $\{y_1 \to f(x_1)\}$
3	$q(f(c), f(x_1))$	*binary resolvent of 2, 2.5 using mgu* $\{y_2 \to c, x_2 \to x_1\}$
4	$\neg q(f(f(x_3)), f(c))$	*variant of* E
5	$\neg q(f(c), f(x_3))$	*binary resolvent of 2, 4 using mgu* $\{y_2 \to f(x_3), x_2 \to c\}$
6	$\square$	*binary resolvent of 3,5 using mgu* $\{x_3 \to x_1\}$

Task 6

Let G be the modal formula $\Diamond A \vee \Box\Box\neg A$. Prove or refut:

1. G is valid in every transitive frame.
2. If $\mathcal{F} \models G$ for a frame $\mathcal{F}$, then $\mathcal{F}$ is transitive.

Solution. *Formula G is valid in every transitive frame. Consider an arbitrary model of an arbitrary transitive frame. Furthermore, let w be any world of the frame. If $\Diamond A$ does not hold in w, then A does not hold in any world that w can see. Now, assume that $\Box\Box\neg A$ does not hold in w. Then $\neg\Box\Box\neg A \equiv \Diamond\neg\Box\neg A \equiv \Diamond\Diamond A$ holds in w. Hence, w can see a world w' that can see a world w'' that can see a world w_A in which A holds. However, because the frame is transitive, w can also see w_A, contradicting the assumption that $\Diamond A$ does not hold in w.*

Furthermore, if the formula is valid in a frame, the frame must be transitive. Consider an arbitrary non-transitive frame $\mathcal{F}$. There are worlds w_1, w_2, w_3 such that $w_1 R w_2, w_2 R w_3$ but not $w_1 R w_3$. Consider the interpretation that sets p to false in all neighbors of w_1, p to true in all remaining worlds. Because p does not hold in all worlds that w_1 can see, $\Diamond p$ is clearly false in p. However, w_2 can see w_3 in which p holds, hence, $\Box\neg p$ is false in w_2. But then $\Box\Box\neg p$ is false in w_1, and the formula is not fulfilled in w_1.

Task 7

Use the proof of the ADRF theorem to show that the function $3 \cdot g(f(x), f(x+1))$ is arithmetically definable, if f and g are arithmetically definable. *Hint:* Specify the witnessing formula, following slide 23 of the last set of slides (on incompleteness).

Solution. *Let G, F be the witnessing formulas of f, g. Then the witnessing formula of the function is simply:*

$$\exists v_f, v_{f'}, v_g(F(v_1, v_f) \wedge F(s(v_1), v_{f'}) \wedge G(v_f, v_{f'}, v_g) \wedge v_2 = 3 \cdot v_g)$$

4 Exam 20.01.2021

Task 1

Formalize the following sentence in classical logic

"If not everybody gets a covid vaccine, there is at least someone who does not get a covid vaccine."

Exhibit either a sequent calculus or a natural deduction derivation for the resulting formula (in case it is valid) or an interpretation that falsifies it (in case it is not valid).

Solution. *We will use the following symbols with the intended meaning:*

$$V(x) \ldots x \text{ gets a covid vaccine.}$$

The formula is then

$$\neg\forall x V(x) \rightarrow \exists x \neg V(x).$$

We prove this using natural deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\exists x V(x)] \qquad \dfrac{[V(a)]}{\exists x V(x)}}{\bot}}{\exists x V(x)} \qquad [V(y)]}{V(y)}}{\forall x V(x)} \qquad [\neg\forall x V(x)]}{\bot}}{\neg V(a)}}{\exists x \neg V(x)}}{\neg\forall x V(x) \rightarrow \exists x \neg V(x)}$$

and sequent calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{V(x) \vdash V(x)}{\vdash V(x), \neg V(x)}}{\vdash V(x), \exists x \neg V(x)}}{\vdash \forall x V(x), \exists x \neg V(x)}}{\neg\forall x V(x) \vdash \exists x \neg V(x)}}{\vdash \neg\forall x V(x) \rightarrow \exists x \neg V(x)}$$

Task 2

Consider the following formulas of classical logic:

(a) $\forall x \forall y (P(x,y) \rightarrow P(y,x))$

(b) $\forall x \forall y \forall z (P(x,y) \wedge P(x,z) \rightarrow P(x,z))$

(c) $\forall x P(x,x)$

Does (a),(b)$\models$(c) holds? Motivate your answer.

Solution. *The consequence claim does not hold. Evidently, (a) described a symmetric relation, and (b) claims that it is also transitive. However, as well all know, that does not make it reflexive.*

Task 3

Let A be the set

$$\{x \mid \Phi_x(x) \downarrow \text{ and } \Phi_x(x) > 2\}$$

Is A recursive, r.e. but not recursive, or none of them? (provide a formal proof)

Solution. *The set is not recursive. For this, define*

$$f(x,z) = \begin{cases} 3 \textit{ if } \Phi_x(x) \downarrow \\ \uparrow \textit{ otherwise.} \end{cases}$$

, and let $\Phi_i \simeq f$. It is easy to see that f is computable. By the s-m-n theorem, we have

$$\Phi_i(x,z) \simeq \Phi_{s_1^1(i,x)}(z).$$

Now, assume that the set was recursive. Then, the characteristic function f' of it is computable. However, this allows us to solve a variation of the halting problem. On input x, we can decide whether $\Phi_x(x) \downarrow$ by computing $f'(s_1^1(i,x))$. If $\Phi_x(x) \downarrow$, then $\Phi_{s_1^1(i,x)}$ terminates on every input, and outputs 3 on every input, hence, f' will return 1. On the other hand, if $\Phi_x(x) \uparrow$, then $\Phi_{s_1^1(i,x)}$ does not terminate, and f' outputs 0. This allows us to decide an undecidable problem, proving that the function cannot be recursive.

The set is however r.e.: It is semi-decidable. On input x, we can compute $\Phi_x(x)$, and if the output is larger than 2, we terminate and output 1. If the output is ≤ 2, we go into an infinite loop.

Task 4

Let I be an infinite set. Prove the following statement "I is recursively enumerable if and only if $I = R(f)$ for some f that is total, computable, and 1-1" (Recall that $R(f)$ denotes the range of the function f.)

Solution. $\Leftarrow$*: Trivial.*

$\Rightarrow$*: Assume that I is r.e. Hence, there exists a total computable function g such that $I = R(g)$. We need to provide a function f that is, in addition, $1-1$. We sketch how this could be done: On input a natural number x, do not simply output $g(x)$, but, output $g(x')$ for the smallest x' such that $g(x')$ is not output by g on input any number from 0 to $x-1$. It is clear that this function is $1-1$. Moreover, we know that the set I is infinite. Hence, for any element $x \in I$, there exists a smallest $x' \in \mathcal{N}$ such that $g(x') = x$. We will eventually reach this x', showing that $R(g) = I$, and furthermore, that the function is computable.*

Task 5

Let G be the modal formula $\Diamond\Box A \vee \Diamond\neg A$. Prove or refute:

1. G is valid in every reflexive frame.
2. If $\mathcal{F} \models G$ for a frame $\mathcal{F}$, then $\mathcal{F}$ is reflexive.

Solution. *Let w be an arbitrary world of an arbitrary frame under an arbitrary variable assignment.*

Case 1: $\Box A$ holds in w. Then, because w can see itself, also $\Diamond\Box A$ holds.

Case 2: $\Box A$ does not hold in w. Then, $\neg A$ holds in w, and it follows that $\Diamond\neg A$ also holds because the frame is reflexive.

However, there are non-reflexive frames in which the formula is valid. The next figures illustrate exactly such a frame.

Indeed, if $\neg A$ holds in w_2, then, since both w_1 and w_2 can see w_2, we know that $\Diamond\neg A$ holds in both worlds as well.

If A holds in w_2, then $\Box A$ holds in w_2 as well. But then we have $\Diamond\Box A$ in both w_1 and w_2, because both of these worlds can see w_2.

Task 6

Show by Robinson-resolution the the clause set $\{C, D, E\}$ is unsatisfiable, where

$$\begin{aligned} C &= \neg q(f(x), y) \vee \neg q(y, f(z)), \\ D &= q(x, y) \vee q(y, x) \vee p(x, g(z)), \\ E &= \neg p(x, g(x)). \end{aligned}$$

Specify all used factors, MGUs, and unified literals.

Solution. *We provide a Robinson-refutation. For convenience, we describe used factors in "half-steps". If no such half-step is given, the factor can be assumed to be the whole formula.*

1	$q(x_1, y_1) \vee q(y_1, x_1) \vee p(x_1, g(z_1))$	*Variant of D*
2	$\neg p(x_2, g(x_2))$	*Variant of E*
3	$q(x_2, y_1) \vee q(y_1, x_2)$	*Resol. 1 and 2 using mgu* $\{z_1 \to x_2, x_1 \to x_2\}$
4	$\neg q(f(x_3), y_3) \vee \neg q(y_3, f(z_3))$	*variant of 4*
4.4	$\neg q(f(z_3), f(z_3))$	*Factor of 4 using mgu* $\{x_3 \to z_3, y_1 \to f(z_3)\}$
4.5	$q(x_2, x_2)$	*Factor of 3 using mgu* $\{y_1 \to x_2\}$
5	$\Box$	*Resol. of 4.4 4.5 using mgu* $\{x_2 \to f(z_3)\}$

Task 7

Use the proof of the ADRF theorem to prove that the formula $f(g(x), 2) + 3$ is arithmetically definable, if f and g are arithmetically definable. *Hint:* Specify the witnessing formula, following slide 24 of the last set of slides (on incompleteness).

Solution. *Let G, F be the witnessing formulas of f, g. Then we have the witnessing formula*

$$\exists v_g, v_f(G(v_1, v_g) \wedge F(v_g, ss0, v_f) \wedge v_2 = sssv_f)$$

5 Exam 31.01.2023

Task 1

Let $Q_n (n > 1)$ be the following formula

$$\forall x_1 \dots \forall x_n \bigvee_{i \neq j} x_i = x_j$$

Provide a formal proof that answers the following question:

- If the interpretation $(\mathcal{A}, \mathcal{E}^{\mathcal{A}})$ is a model for Q_n which is the size of its domain (i.e. $|D_{\mathcal{A}}|$)?

Solution. *Let $(\mathcal{A}, \mathcal{E}^{\mathcal{A}})$ be a model of Q_n. Then, $|D_{\mathcal{A}}| < n$. Assume otherwise, in particular, assume there exist $n+$ distinct elements $d_1, \dots, d_n \in D_{\mathcal{A}}$. By the semantics of the all quantifier, this implies that*

$$v^{\mathcal{A}}_{\mathcal{E}^{\mathcal{A}}[x_1/d_1][x_2/d_2]\dots[x_n/d_n]}(\bigvee_{i \neq j} x_i = x_j) = 1$$

However, since all the elements are distinct, this cannot be the case. On the other hand, if the domain has size less than n, it is clear that any variable-assignment will always assign the same domain element to two different variables of the disjunction, and hence, it will be fulfilled.

Task 2

Formalize the following sentence in classical logic: If there are two students that pass the exam, then all students pass it.

Exhibit either a sequent calculus or a natural deduction derivation for the resulting formula (in case it is valid) or an interpretation that falsifies it (in case it is not valid).

Solution. *We use the following symbols with the intended meaning:*

$$P(x) \dots x \text{ passes the exam.}$$

The formula is then:

$$\exists s_1 \exists s_2 (P(s_1) \wedge P(s_2) \wedge s_1 \neq s_2) \to \forall x (P(x)).$$

We assume that we have a logic with equality at hand.

The formula is not valid. Consider the structure $\mathcal{A}$, with $D_{\mathcal{A}} = \{0, 1, 2\}$, and $P^{\mathcal{A}} = \{0, 1\}$. It is easy to see that the left side of the implication will be true, as there are two distinct domain elements for which the predicate $P^{\mathcal{A}}$ is true, however, the right side is not, as the predicate is false for domain element 2.

Task 3

Is the set $\{x \in \mathcal{N} \mid \neg\exists y \text{ s.t. } \Phi_x(y) = \Phi_x(y+2)\}$ recursive, r.e. or none of them? (Motivate your answer)

Solution. *The set is extensional, because the property is solely a property of a function. Furthermore, there are clearly functions for which this holds (e.g. the identity function), and functions for which this does not hold (e.g. the constant 0 function). Thus the set is not recursive by Rice theorem.*

Furthermore, the set is not r.e. We show this by proving that the complement is r.e., and then the claim follows by Post theorem. It is easy to see that the complement is recursive, as we can use dove-tailing to find an input y such that $\Phi_x(y) = \Phi_x(y+2)$ if it exists.

Task 4

If I is a recursive set and J is recursively enumerable, what can be said about $I \setminus J = \{x \mid x \in I \text{ and } x \textit{ not} \in J\}$ in the following cases:

(a) J is recursive,

(b) I is finite.

Solution. *(a) If both J and I are recursive, then $I \setminus J$ is clearly recursive, on input an element x, we can just check whether it is in I but not in J.*

(b) If I is finite, then $I \setminus J$ must be recursive, because it is a subset of a finite set, and hence one of finitely many possible sets.

Task 5

Compute all Robinson-resolvents of the following two clauses C and D:

$$C = P(h(x,u),y) \vee \neg P(x, f(y)) \vee \neg P(u,y)$$
$$D = P(f(f(x),b)) \vee P(y,x)$$

In each case, specify the used factors, MGUs, and unified literals. (b is a constant.)

Solution. *Let us compute variable-disjoint variants first for convenience.*

$$C = P(h(x_1,u_1),y) \vee \neg P(x_1, f(y_1)) \vee \neg P(u_1,y_1)$$
$$D = P(f(f(x_2),b)) \vee P(y_2,x_2)$$

We first compute all factors of C. The first factor here is C itself. There are no further factors, since no literals of C can be unified.

Next, we compute the factors of D. These are D itself, and furthermore, we can obtain the factor $P(f(f(b)),b)$ by using the mgu $\{y_2 \to f(f(b)), x_2 \to b\}$.

First, we consider the resolvents of the factors C, D. We can unify $P(f(f(x_2)),b)$ with $P(u_1,y_1)$ using mgu $\{u_1 \to f(f(x_2)), y_1 \to b\}$. This results in the resolvent $P(h(x_1, f(f(x_2))), b) \vee \neg P(x_1, f(b)) \vee P(y_2, x_2)$.

We can unify $P(y_2,x_2)$ with $P(u_1,y_1)$ using mgu $\{u_1 \to y_2, y_1 \to x_2\}$. The resolvent is then $P(h(x_1,y_2),y) \vee \neg P(x_1, f(x_2)) \vee \vee P(f(f(x_2),b))$.

Finally, we consider the resolvents of the factor $P(f(f(b)),b)$ with C. We can unify with $P(u_1,y_1)$ using the mgu $\{u_1 \to f(f(b)), y_1 \to b\}$. The resulting resolvent is $P(h(x_1, f(f(b))), y) \vee \neg P(x_1, f(b))$.

Task 6

Let G be the modal formula $A \lor B \lor \neg\Diamond(\Box A \lor \Box B)$. Prove or refute:

1. G is valid in every symmetric frame.
2. If $\mathcal{F} \models G$ for a frame $\mathcal{F}$, then $\mathcal{F}$ is symmetric.

Solution. *Let $\mathcal{F}$ be a symmetric frame. Let $\mathcal{M} = (W, R, V)$ be any model based on $\mathcal{F}$, and $w \in W$. If $v_{\mathcal{M}}(A, w) = 1$ or $v_{\mathcal{M}}(B, w) = 1$, then the formula holds in w. So, consider the case that $v_{\mathcal{M}}(A, w) = v_{\mathcal{M}}(B, w) = 0$. Towards a contradicting, assume that $v_{\mathcal{M}}(\neg\Diamond(\Box A \lor \Box B), w) = 0$, and hence, $v_{\mathcal{M}}(\Diamond(\Box A \lor \Box B), w) = 1$. Then $\exists w' \in W, wRw'$ and $v_{\mathcal{M}}(\Box A \lor \Box B, w') = 1$. However, as wRw', and the frame is symmetric, we have $w'Rw$. This contradicts that $v_{\mathcal{M}}(\Box A \lor \Box B, w') = 1$, because both A and B are not true in w.*

For the other direction, let (W, R) be an arbitrary non-symmetric frame. Since the frame is non-symmetric, $\exists w_1, w_2 \in W$ such that $w_1 R w_2$ but not $w_2 R w_1$. Let V be the assignment that sets p to false in w_1, true everywhere else, and let $\mathcal{M} = (W, R, V)$. Then, $v_{\mathcal{M}}(p, w_1) = 0$, and furthermore, $v_{\mathcal{M}}(p, \Diamond(\Box p \lor \Box p), w_1) = 1$, showing that the formula is not valid in such a frame. Regarding the last point: We know that $v_{\mathcal{M}}(\Box p, w_2) = 1$, because p holds in all worlds except w_1, which is not seen by w_2. It then follows that $\Diamond(\Box p \lor \Box p)$ actually holds in w_1, as $w_1 R w_2$.

Task 7

Use the proof of the ADRF theorem to show that the function $f(x+1, g(x, 2y)) + z$ is arithmetically definable, if f, g are arithmetically definable by explicitely stating the witnessing formula.

Solution. *Let F, G be the witnessing formulas of f, g, respectively.*

Then the witnessing formula we are seeking is

$$\exists v_g, v_f, v_2 (v_2 = 2 \cdot y \land G(v_1, v_2, v_g) \land F(sv_1, v_g, v_f) \land v_4 = v_f + v_3).$$

6 Exam 21.03.2021

Task 1

Formalize the following sentences:

(a) For every unfortunate there is some goat that bites him.

(b) Alf is unfortunate.

(c) Bic is an unfortunate goat.

(d) Bic bites Alf.

Does (a),(b),(c) $\models$ (d)? Motivate your answer.

Solution. *We use the following symbols with the intended meaning:*

$U(x) \ldots x$ *is unfortunate*

$G(x) \ldots x$ *is a goat*

$B(x, y) \ldots x$ *bites* y

$alf \ldots$ *Alf*

$bic \ldots$ *Bic*

Then, the formulas are:

(a) $\forall x(U(x) \rightarrow \exists y(G(y) \wedge B(y, x)))$

(b) $U(alf)$

(c) $U(bic) \wedge G(bic)$

(d) $B(bic, alf)$

It is not the case that (a),(b),(c) $\models$ (d). Consider $\mathcal{A}$ with $D_{\mathcal{A}} = \{0,1\}$ and $alf^{\mathcal{A}} = 0, bic^{\mathcal{A}} = 0, U = \{0\}, G = \{0,1\}, B = \{(1,0)\}$. Evidently, this structure fulfills (a)-(c), but not (d). Intuitively, it is clear that this consequence is not correct: Just because Alf is unfortunate and Bic a goat, that does not mean that Bic bites Alf, as Alf could be bitten by any other goat.

Task 2

Describe the interpretations that are models for the formula

$$\forall x \exists y(\neg(x = y) \wedge A(x, y))$$

where $=$ is the equality predicate.

Solution. *Such an interpretation $(\mathcal{A}, \mathcal{E}^{\mathcal{A}})$ requires at least two domain elements. Furthermore, for any domain element x, there must exist another domain element y such that $A^{\mathcal{A}}(x, y)$ holds. Graphically, we could express the domain element as vertices, and the A-predicate as arcs. Then the resulting graph would have no vertex without an outgoing edge.*

Task 3

Is the countable union of recursive sets recursive? (Provide a proof or a counterexample)

Solution. *Let $\mathcal{A} := \{A_1, A_2, \dots\}$ be a countable set of recursive sets. Then, $\bigcup_{A \in \mathcal{A}} A$ is not recursive. Let*

$$A_k = \begin{cases} \{k\} & \textit{if } \Phi_k(k) \uparrow, \\ \emptyset & \textit{otherwise.} \end{cases}$$

Then, it is clear that A_k is recursive for any k, as it is finite. Then $\bigcup_{k \in \mathcal{N}} A_k$ is a countable union of recursive sets. However, if the resulting set was recursive, we could decide an undecidable problem using the characteristic function, which cannot be.

Task 4

Let Φ_i be a partial computable function and consider the following sets:

1. $K = \{x \mid x \notin \mathit{Ran}(\Phi_i)\}$
2. $J = \{x \mid x \in \mathit{Dom}(\Phi_i) \wedge x \in \mathit{Ran}(\Phi_i)\}$

Are they recursive, r.e., or neither? (Ran = Range and Dom = Domain ($= \{x \mid \Phi_i(x) \downarrow\}$))

Solution. *Set K is not recursive, and not r.e in general. For example, if Φ_i is the identity function, K is clearly recursive.*

First, we show that it is not recursive in general. For this purpose, let

$$\Phi_i(x) := \begin{cases} x & \textit{if } \phi_x(x)\downarrow, \\ \uparrow & \textit{otherwise.} \end{cases}$$

It is easy to see that this function is computable. If K was recursive for Φ_i, we could clearly decide an undecidable problem, which cannot be.

On the other hand, the complement of K is r.e. We can equivalently show that it is semi-decidable. Consider the following algorithm:

- *Take integer x as input*
- *Use dove-tailing to try out all inputs for any finite amount of time*
- *If some output is x, output 1*

It is easy to see that this sketch of an algorithm can be turned into a proper algorithm, that outputs 1 when x is in the range of Φ_i, and does not terminate otherwise. By Post Theorem, K is not r.e.

For J, observe again that it can be recursive for specific Φ_i. But it is not recursive in general. Consider the function Φ_i from above, we have $Dom(\Phi_i) = Ran(\Phi_i)$, and hence, $J = K$ is not recursive in this case. However, J is r.e. We sketch a proof for J being semi-decidable. Consider the following algorithm:

- *Take integer x as input*
- *Run $\Phi_i(x)$*
- *Output 1*

By executing this algorithm, and the algorithm from above in parallel, we have a semi-decidable procedure to determine whether an input x is in both the domain and the range of Φ_i.

Task 5

Show by Robinson-resolution that the clause set $\{C, D, E\}$ is unsatisfiable, where

$$\begin{aligned} C &= \neg p(f(x), y) \vee p(f(y), f(x)), \\ D &= p(x, a) \vee p(f(y), a), \\ E &= \neg p(f(f(x)), f(a)). \end{aligned}$$

Specify all used factors, MGUs, and unified literals. (a is a constant, x, y are variables.)

Solution. *We begin by computing variable-disjoint variants:*

$$\begin{aligned} C &= \neg p(f(x_1), y_1) \vee p(f(y_1), f(x_1)), \\ D &= p(x_2, a) \vee p(f(y_2), a), \\ E &= \neg p(f(f(x_3)), f(a)). \end{aligned}$$

Next, we specify a factor of D, for all the other used factors, assume it is the trivial factor.

$$F = p(f(y_2), a) \textit{ unifying both literals of } D \textit{ using mgu } \{x_2 \to f(y_2)\}.$$

Now, we can give the Robinson-refutation:

1.	$p(x_2, a) \lor p(f(y_2), a)$	*Formula D*
2.	$\neg p(f(x_1), y_1) \lor p(f(y_1), f(x_1))$	*Formula C*
3.	$\neg p(f(f(x_3)), f(a))$	*Formula E*
4.	$p(f(a), f(y_2))$	*resol. literals of F,2. using* $\{x_1 \to y_2, y_1 \to a\}$
5.	$\neg p(f(a), f(x_3))$	*resol. literals of 3, 2 using* $\{y_1 \to f(x_3), x_1 \to a\}$
6.	$\Box$	*resol. 4. 5. using* $\{y_2 \to x_3\}$

Task 6

Let G be the modal formula $\neg\Diamond\Box A \lor A$. Prove or refute:

1. G is valid in every symmetric frame.
2. If $\mathcal{F} \models G$ for a frame $\mathcal{F}$, then $\mathcal{F}$ is symmetric.

Solution. *Let $\mathcal{M} = (W, R, I)$ be an arbitrary interpretation based on a symmetric frame. Consider $w \in W$. If $w_{\mathcal{M}}(A, w) = 1$ then clearly also $w_{\mathcal{M}}(\neg\Diamond\Box A \lor A, w) = 1$. Otherwise, $w_{\mathcal{M}}(A, w) = 0$. Consider arbitrary w' such that wRw'. Because then also $w'Rw$, and A is false in w, we know that $w_{\mathcal{M}}(\Box A, w') = 0$. Hence, no world visible from w fulfills $\Box A$, and it follows that $w_{\mathcal{M}}(\neg\Diamond\Box A) = 1$, concluding the proof.*

Consider arbitrary non-symmetric frame (W, R). Then, $\exists w_1 w_2 \in W$ such that $w_1 R w_2$ but not $w_2 R w_1$. If we set p to true in all worlds except for w_1, then p is false in w_1. However, since w_2 does not see w_1, $\Box p$ is clearly true in w_2. Hence, $\Diamond\Box p$ is true in w_1, and $\neg\Diamond\Box p$ is false in w_1. It follows that under this interpretation, $\neg\Box p \lor p$ is false in w_1. So, if G is valid in a frame, the frame must be symmetric.

Task 7

Use the proof of the ADRF theorem to show that the function $f(x + 2, g(x, x))$ is arithmetically definable, if f, g are arithmetically definable by explicitly stating the witnessing formula.

Solution. *Let F, G be the witnessing formulas of f, g, respectively.*

Then the witnessing formula we are seeking is

$$\exists v_g (G(v_1, v_1, v_g) \land F(ssv_1, v_g, v_2)).$$

7 Exam 30.05.2017

Task 1

Prove in sequent calculus

$$(A \to \exists x B(x)) \to \exists x (A \to B(x))$$

where x is not free in A.

Solution. *We provide a derivation:*

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A, B(a)}{\vdash A, A \to B(a)}}{\vdash A, \exists x(A \to B(x))} \qquad \dfrac{\dfrac{\dfrac{B(a), A \vdash B(a)}{B(a) \vdash A \to B(a)}}{B(a) \vdash \exists x(A \to B(x))}}{\exists x B(x) \vdash \exists x(A \to B(x))}}{(A \to \exists x B(x)) \vdash \exists x(A \to B(x))}}{\vdash (A \to \exists x B(x)) \to \exists x(A \to B(x))}$$

Task 2

Formalize the sentence "every number greater than 1 is divisible by a prime number" using the symbols = (equality predicate), < (predicate strictly less than), 1 (constant 1) and the binary predicate | (divisible). [Note that you dont have a predicate prime number at your disposal]

Solution. *The formula is simply*

$$\forall x((\neg(x < 1) \wedge \neg(x = 1)) \to \exists y(P(y) \wedge y \mid x))$$

where

$$P(y) \coloneqq \forall z(z \mid y \to ((z = y) \vee (z = 1))).$$

Task 3

Is the following set

$$\{x \mid \neg\exists y \phi_x(y) \downarrow\}$$

Recursive, r.e. or none of them? (Motivate your answer)

Solution. *This set is extensional (note that the property of terminating for some input is clearly a property of the function itself). Furthermore, there exists a function (e.g. produced by a looping Turing machine) that is in the set, and a function (e.g. the constant 0 function) that is not in the set. By Rice theorem, the set is thus not recursive.*

Furthermore, the set is not r.e., which we show by showing that the complement is r.e. Then, the initial set cannot be r.e. due to Post theorem. We show the equivalent statement that the complement is semi-decidable. Consider the following algorithm: It is easy to see that, whenever

```
Input: integer x
k := 0
while True do
    (i, t) := (α1(k), α2(k))
    Let Φx(i) run for at most t steps
    if The computation terminated in these t steps then
        return 1
    end
    k := k + 1
end
```

an element x is in the complement (and hence, it terminates for some input), then, we correctly output 1. *Otherwise, the algorithm loops forever.*

Task 4

Let I be any set of indexes of computable functions. Consider the following statement: "I is infinite if and only if I is extensional." Is the statement true? (Motivate your answer and argue separately for each of the two cases: "if" and "only if".)

Solution. *The statement is false.*

$\Rightarrow$*:*

If I is infinite, then it is not necessarily extensional. E.g. $\{x \mid x \text{ is even}\}$ *is infinite, but not extensional.*

$\Leftarrow$*: If I is extensional, it is infinite because of the Padding Lemma.*

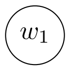
Task 5

Prove or refute: $(\neg B \vee \Diamond A) \rightarrow (B \rightarrow A)$ characterizes the reflexivity of Kripke frames. Treat the two directions of the claim separately and argue directly about the corresponding frames.

Solution. *First, we show that the formula is not valid in all reflexive frames. Consider the following interpretation:*

Then, the formula $(\neg q \vee \Diamond p) \rightarrow (q \rightarrow p)$ *is not true in world* w_1 *under this interpretation, yet, the underlying frame is reflexive. Regarding the formula not holding: Since p is true in* w_2 *and* $w_1 R w_2$*, clearly* $\Diamond p$ *is true in* w_1*, and hence* $(\neg q \vee \Diamond p)$ *is true in* w_1*. Furthermore, q holds in* w_1 *as well, but p does not. Hence,* $q \rightarrow p$ *is false in* w_1*, making the overall formula false in* w_1 *as well.*

Furthermore, the formula is valid in some non-reflexive frames:



Clearly, in w_1*,* $\Diamond A$ *is false regardless of formula A because* w_1 *cannot see any world. So, if the assumption of the implication is true, we know that* $\neg B$ *holds in* w_1*. But then the second implication is trivially true, and the overall formula holds.*

Task 6

Compute all factors of the clause $p(x, f(x)) \vee p(f(y), y) \vee p(f(a), z)$. Specify the MGU and the unified literals for each factor. (x, y, z are variables, a is a constant.)

Solution. *The first factor is the trivial factor, the clause itself.*

For the second factor, observe that we can unify $p(x, f(x))$ *and* $p(f(a), z)$ *using mgu* $\{x \rightarrow f(a), z \rightarrow f(f(a))\}$*. The resulting factor is* $p(f(a), f(f(a))) \vee p(f(y), y)$*.*

For the third factor, we can unify $p(f(y), y)$ *and* $p(f(a), z)$ *using mgu* $\{y \rightarrow a, z \rightarrow a\}$*. The resulting factor is* $p(x, f(x)) \vee p(f(a), a)$*.*

Because we cannot unify $p(x, f(x))$ *and* $p(f(y), y)$*, there are no further factors.*

Task 7

Show: The diagonal set U^* of the set U of Gödel numbers of all expressions that are not true sentences ($\notin \mathcal{T}$) is not expressible in any arithmetic system.

Solution. *Assume that U^* is expressible. By Lemma D, this means that there exists a Gödel sentence S for U. Per definition of Gödel sentences, this means that*

$$S \in \mathcal{T} \Leftrightarrow \ulcorner S \urcorner \in U.$$

Which is nonsensical: it cannot be the case that S is true, and yet its Gödel number is in the set of Gödel numbers of expressions that are not true.

8 Exam 27.06.2016

Task 1

(1.1) Formalize the following argument in predicate logic "If no showman is sincere and there is no politician that is not a showman, then all politicians are not sincere."
and (1.2) prove or disprove its validity.

Solution. *We use the following symbols with the intended meaning:*

$$\begin{aligned}
&S(x) \ldots x \textit{ is a showman},\\
&P(x) \ldots x \textit{ is a politician},\\
&C(x) \ldots x \textit{ is sincere}.
\end{aligned}$$

The formula is then

$$(\forall x(S(x) \to \neg C(x)) \land \forall x(P(x) \to S(x))) \to \forall x(P(x) \to \neg C(x)).$$

Which is indeed valid. To see this, consider an arbitrary interpretation $I = (\mathcal{A}, \mathcal{E}^{\mathcal{A}})$, and assume that the formula does not hold under this interpretation. Then, $v^I(\forall x(P(x) \to \neg C(x))) = 0$, and $v^I((\forall x(S(x) \to \neg C(x)) \land \forall x(p(x) \to S(x))) = 1$. Hence, there is an element $d \in D_{\mathcal{A}}$ such that $d \in P^{\mathcal{A}}$, and furthermore, $d \in C^{\mathcal{A}}$. By the conjunction, if an element $x \in P^{\mathcal{A}}$, then also $x \in S^{\mathcal{A}}$. Hence, we can conclude that $d \in S^{\mathcal{A}}$. But, the conjunction also states that, if an element $x \in S^{\mathcal{A}}$, then $x \notin C^{\mathcal{A}}$. Hence, we obtain $d \notin C^{\mathcal{A}}$, which contradicts the assumption.

Task 2

Is the set

$$I := \{x \in \mathcal{N} \mid \neg \exists y \Phi_x(y) \downarrow\}$$

recursive, r.e. or none of them? Motivate your answer.

Solution. *See section 7.*

Task 3

Is the following function

$$f(x) = \begin{cases} 1 & \text{if } \Phi_x(x+1)\downarrow \text{ and } x \leq 50 \\ 0 & \text{otherwise} \end{cases}$$

computable? Provide either an informal algorithm (in case the function is computable) or a formal proof that such algorithm cannot exist.

Solution. *Function f is computable. Consider the following "non-constructive" algorithm:*

```
Input: integer x
if x > 50 then
  | return 0
else
  | return Φ_x(x + 1) ↓
end
```

It is easy to see that the algorithm correctly computes the function. Furthermore, we can simply hardcode whether $\Phi_x(x+1)\downarrow$ for the case where $x \in [0 \ldots 50]$, and hence, the algorithm always terminates.

Task 4

Compute *all* Robinson-Resolvents of the two clauses $p(f(x,y),y) \vee p(z,a)$ and $\neg p(f(x,x),y)$.

Solution. *We compute variable disjoint variants $A \coloneqq p(f(x_1,y_1),y_1) \vee p(z_1,a)$ and $B \coloneqq \neg p(f(x_2,x_2),y_2)$.*

Now, let us compute factors of these clauses. The only factor is from A using $\{y_1 \to a, z_1 \to f(x_1,a)\}$, and it is $F \coloneqq p(f(x_1,a),a)$.

We only have factor F and the trivial factors.

We can resolve upon $p(z_1,a)$ of A and $\neg p(f(x_2,x_2),y_2)$ using $\{z_1 \to f(x_2,x_2), y_2 \to a\}$. This results in resolvent $p(f(x_1,y_1),y_1)$.

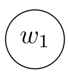
We can resolve upon $p(f(x_1,y_1),y_1)$ and $\neg p(f(x_2,x_2),y_2)$ using $\{x_1 \to x_2, y_1 \to x_2, y_2 \to x_2\}$. The resolvent is $p(z,a)$.

We can resolve upon $p(f(x_1,a),a)$ from F and $\neg p(f(x_2,x_2),y_2)$ using $\{x_2 \to a, x_1 \to a, y_2 \to a\}$. The resolvent is $\square$.

Task 5

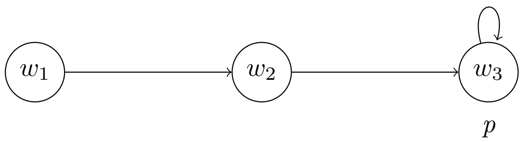
1. Does $\mathcal{F} \models \Diamond A \vee \neg\Diamond\square A$ entail the seriality of $\mathcal{F}$?
2. Does the seriality of $\mathcal{F}$ entail $\mathcal{F} \models \Diamond A \vee \neg\Diamond\square A$?

Solution. *1 is not true. There are frames on which the formula is valid that are not serial:*



It is clear that $\Diamond B$ is false in w_1 under any variable assignment. Hence, $\neg\Diamond\Box A$ is true in w_1, and so is the whole formula.

2 is not true. There are serial frames in which the formula is not valid. Consider the following interpretation:



It is clear that the underlying frame is serial. However, $\Diamond p$ does not hold in w_1, because p does not hold in w_2, the only world that w_1 can see. However, w_2 can only see w_3, and p holds in w_3. Hence, $\Box p$ holds in w_2, and $\Diamond\Box p$ holds in w_1. Thus, the formula $\Diamond p \vee \neg\Diamond\Box p$ is not valid in this frame.

Task 6

Is the set of Gödel numbers of sentences that are *not true* in the standard model of arithmetic recursively enumerable? Explain!

Solution. *Assume the set was r.e. Then, we can refute every expression that is false, as e.g. the execution trace of the characteristic function shows. Consider an expression S that is true bot not provable. Then we could prove $\neg S$ being false, however, that is essentially a proof of S. Since such a expression S does indeed exist by incompleteness, we know that the set cannot be r.e.*

9 Exam 09.07.2014

Task 1

Provide or provide a counter example for the following statement:

- $Q(x) \to \forall x P(x)$ is logically equivalent to $\forall x(Q(x) \to P(x))$

Solution. *The formulas are not logically equivalent. Consider the interpretation $(\mathcal{A}, \mathcal{E}^{\mathcal{A}})$ with $D_{\mathcal{A}} = \{0,1\}$, $P_{\mathcal{A}} = \emptyset, Q_{\mathcal{A}} = \{1\}$ and $\mathcal{E}^{\mathcal{A}}(x) = 0$. Then, this interpretation satisfies only one of the two formulas.*

Task 2

Provide a formal proof of the following statement in the classical sequent calculus LK:

- $\forall x(A(x) \to B(x)), \forall x A(x) \models \forall B(x)$

Is your proof also a valid LJ proof?

Solution. *We provide a derivation. The derivation is not a valid LJ proof, because we apply $(\to L)$.*

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a)\vdash A(a),B(a) \qquad B(a),A(a)\vdash B(a)}{A(a)\to B(a),A(a)\vdash B(a)}}{\forall x(A(x)\to B(x)),A(a)\vdash B(a)}\to L}{\forall x(A(x)\to B(x)),\forall x A(x)\vdash B(a)}\forall L}{\forall x(A(x)\to B(x)),\forall x A(x)\vdash \forall B(x)}\forall R$$

Task 3

If I is a recursive set and J is recursively enumerable, what can be said about $I \setminus J = \{x \mid x \in I \text{ and } x \notin J\}$ in the following cases:

(a) $J = \emptyset$,

(b) J is recursive,

(c) J is finite,

(d) J is infinite and not recursive.

Solution. *(a) If $J = \emptyset$, then clearly $I \setminus J = I$, and hence the set is recursive.*

(b) If J is recursive, then $I \setminus J$ is recursive: Given an integer, we can simply check whether it is in I and not in J.

(c) If J is finite, then J is recursive, and hence we are in case (b).

(d) If J is infinite and not recursive, then $I \setminus J$ is not even necessarily r.e. To see this, consider the case that $I = \mathcal{N}$. Then, $I \setminus J = \bar{J}$. However, we know that the complement of a set that is r.e. is not always r.e. Otherwise, any set that is r.e. would also be recursive by Post theorem, but this is clearly not true.

Task 4

Compute all Robinson-Resolvents of the two clauses $\neg p(x, g(y))$ and $p(y, z) \vee p(g(x), y)$.

Solution. *First, compute variable-disjoint variants $C_1 := \neg p(x_1, g(y_1))$ and $C_2 := p(y_2, z_2) \vee p(g(x_2), y_2)$. We proceed to compute all factors. Clearly, the only factor of the fist clause is the clause itself.*

For the second clause, we have the trivial factor, and the factor $F := p(g(x_2), g(x_2))$ obtained using $\{y_2 \to g(x_2), z_2 \to g(x_2)\}$.

Next, we compute all resolvents of C_1 and C_2:

Resolved literals	*Resolvent*	*mgu*
1,2	$p(g(x_2), x_1)$	$\{y_2 \to x_1, z_2 \to g(y_1)\}$
1,3	$p(g(y_1), z_2)$	$\{x_1 \to g(x_2), y_2 \to g(y_1)\}$

And resolvents of C_1 and F:

Resolved literals	*Resolvent*	*mgu*
1,1	$\square$	$\{x_1 \to g(x_2), y_2 \to g(x_2)\}$

Task 5

Prove or refute: $(A \wedge \neg\square B) \vee (\lozenge\neg B \to \neg A)$ is valid in every Kripke frame.

Solution. *The Formula is indeed valid in every frame. Let (W, R, V) be any interpretation. Fix $w \in W$.*

Case 1: A is false in w. Then, $C \to \neg A$ is true in w for any formula C, and hence, the right subformula is true and so is the whole formula.

Case 2: A is true in w. Then, the formula is clearly true if additionally $\neg\square B$ is true in w. So, consider the case that $\neg\square B$ is false in w. Hence, all worlds that w can see have the property that B holds there. Thus, $\lozenge\neg B$ is false in w, and the implication of the formula is clearly true.

Task 6

Suppose that the system Σ over some standard arithemtic language is sound and that for all Turingmachines M the statement 'M terminates on every input' can be expressed in Σ. Can one conclude that Σ is incomplete? Why and how?

Solution. *Towards a contradiction, assume that the system is complete. Furthermore, let E be the expression of the statement above. Then,*

$$E(x) \in \mathcal{T} \leftrightarrow x \textit{ halts on every input} .$$

If the system was complete, we could hence prove that a machine does not hold on any input if that is the case. This contradicts that the halting problem is undecidable.