Name: ______________________________

Registration number: ______________________________

186.835 Mathematical Programming
Final Exam
June 28, 2021

Instructors: Georg Brandstätter, Mario Ruthmair
Duration: 120 min

Question	max. Points	Points
1	7	
2	6	
3	6	
4	8	
5	7	
6	6	
	40	

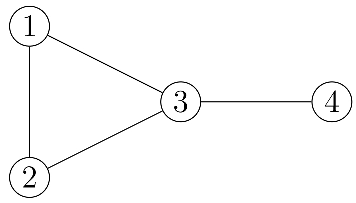
- This is an open book exam which means you can use any resources available, but you have to do it alone.
- Ensure that your solutions and **all necessary intermediate steps** are described / shown in an understandable and readable manner.
- You can do the exercises either electronically (LaTeX, Word, etc.) or on paper.
- Compile all your solutions (electronic documents, scans, photos) into a **single PDF** and **upload it to TUWEL** until the end of the exam.
- If you have any questions during the exam, there is a live Q&A session:
  https://tuwien.zoom.us/j/2409933302

Good Luck!

1. *(7 points) Modeling: Wind Power Farm*

   Several wind turbines should be built in a given area. There are four potential locations $L = \{1, 2, 3, 4\}$, and a wind turbine at location i would lead to 20 units of building costs and produce on average e_i units of energy per day, see the table below. Unfortunately, wake effects lead to a reduction of energy production if wind turbines are too close together. If a wind turbine has one or more neighboring turbines (indicated by edges in the graph below), its produced energy is reduced to 75% of the nominal value e_i. The goal is to choose a subset of locations for building wind turbines that maximizes the total energy output per day such that the total building costs do not exceed a given budget of 50 units.

location i	e_i
1	20
2	24
3	40
4	20



   (a) Formulate this problem as a (mixed) integer linear program. Describe all variables and constraints.

   (b) State an optimal solution including all variables with non-zero values and the corresponding objective value.

2. *(6 points) Lagrangian Relaxation*

Consider the following binary integer program:

$$z = \max \quad 3x_1 + x_2 + x_3 \tag{1}$$
$$2x_1 + 3x_2 + x_3 = 4 \tag{2}$$
$$4x_1 + 2x_2 + 3x_3 \leq 5 \tag{3}$$
$$x_1, x_2, x_3 \in \{0, 1\} \tag{4}$$

(a) Identify the optimal value z and state an optimal solution.

(b) Relax constraint (2) in the usual Lagrangian way and write down the Lagrangian subproblem $z(u)$ depending on the Lagrangian multiplier u associated to the relaxed constraint.

(c) Run 2 iterations of the subgradient algorithm, starting with $u^0 = 0$ and using step size $\mu_0 = 1$ (you do not need to compute u^2). Note that you do not need the $\max$-operator when updating parameter u since it is allowed to get negative. State in each iteration the optimal value of the Lagrangian subproblem and an according optimal solution. If the solution is feasible for the original problem, compute the corresponding original objective value.

(d) What is the computational complexity of the Lagrangian subproblem? Does it have the integrality property? Note that the optimal value of the LP relaxation of the original binary program is 3.375.

3. *(6 points) Vehicle Routing By Branch-and-Cut*

A feasible solution to the capacitated vehicle routing problem in an undirected graph $G = (\{0\} \cup C, E)$ with depot 0, customer nodes $C = \{1,2,3,4,5\}$, edges E, demands d_i for all customer $i \in C$ (see table below), and multiple vehicles each with capacity $Q = 10$ can be characterized as follows:

- Each **customer** is visited exactly once and its demand is fully served.
- All vehicles start and end their tour at the depot 0, i.e., there are no tours disconnected from the depot.
- The sum of all demands in a tour does not exceed the vehicle capacity.

Assume that we run a branch-and-cut approach for the undirected graph below based on the following formulation (omitting the objective function):

$$x(\delta(j)) = 2 \qquad \forall j \in C \tag{5}$$

$$x(\delta(S)) \geq 2\left\lceil \frac{1}{Q}\sum_{i\in S} d_i \right\rceil \qquad \forall S \subseteq C \tag{6}$$

$$x_e \in \{0,1,2\} \qquad \forall e = \{0,j\} \in E, j \in C \tag{7}$$

$$x_e \in \{0,1\} \qquad \forall e = \{i,j\} \in E,\ i,j \in C \tag{8}$$

Capacity constraints (6) are similar to undirected cutset constraints and ensure connectivity and the vehicles' capacity: They state that each subset S of customers needs a minimum number of vehicles to get fully served, i.e., $\left\lceil \frac{1}{Q}\sum_{i\in S} d_i \right\rceil$, and each of those vehicles need to enter and leave set S on 2 different edges. Note that edges incident to the depot can be used twice to allow single-customer tours.

We start only with a subset of the constraints above and in some branch-and-bound node we obtain the LP relaxation solution shown in the graph below: The value next to each edge corresponds to its LP solution value (variable names are mostly omitted for better readability).

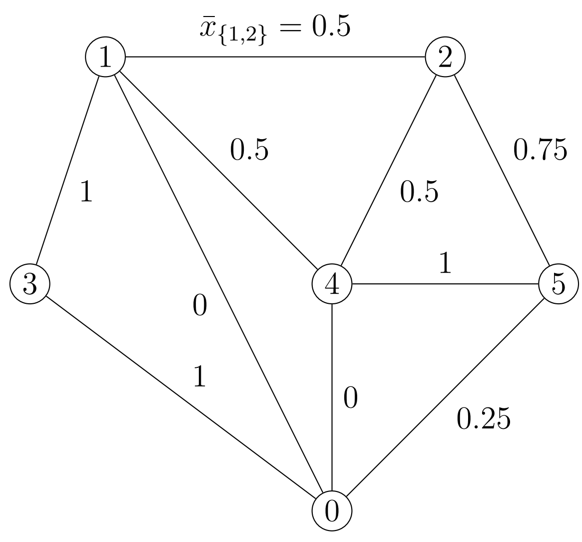
(a) Find at least one **violated** degree constraint, if one exists.

(b) Find at least one **violated** subtour elimination constraint (they are valid for subsets $S \subseteq C$ but not included in the formulation above), if one exists.

(c) How many vehicles do we need at least to serve all customers?

(d) Find at least **two violated** capacity constraints.

customer i	demand d_i
1	4
2	3
3	6
4	2
5	5



4. *(8 points) Modeling: Vaccine Delivery*

The city of Vienna has hired your company to handle the distribution of vaccines to its vaccination centers. Your goal is to accomplish this as cheaply as possible.

Vienna operates a set of vaccination centers $C = \{1, 2, \ldots, n\}$, each of which has a daily demand for d_i boxes of vaccines. Your single delivery vehicle, which can carry up to D boxes, leaves your central warehouse (which is located at location 0) fully loaded in the morning. Driving **from** any location $i \in \{0\} \cup C$ **to** another location $j \in \{0\} \cup C$ takes t_{ij} minutes and costs c_{ij} Euro for fuel. Unloading a single box takes u minutes. To ensure timely opening of all vaccination centers, all vaccines must be delivered and unloaded within T minutes after you leave your warehouse. Your vehicle returns to the warehouse after its final delivery.

Instead of delivering all vaccines yourself, you can subcontract the cryo-logistics company *DeepFreeze* to handle some deliveries for you. *DeepFreeze* guarantees the timely delivery of all necessary vaccines to any vaccination center $i \in C$ for a fixed delivery cost of f_i Euro plus a handling cost of h Euro per box. To simplify logistics at each center, split deliveries are **not** allowed: each center must receive its full complement of vaccines at once, either from you or from *DeepFreeze*.

(a) Formulate this problem as a (mixed) integer linear program. Describe all variables and constraints you used. You may use a compact formulation (polynomially many variables and constraints) or one solved by branch-and-cut.

(b) Describe a **simple** construction heuristic for solving this problem. It must not generate solutions where *DeepFreeze* handles all deliveries, unless that is an optimal solution.

5. *(7 points) Chvátal-Gomory Cutting Planes*

Consider the polyhedron P described by the following linear inequalities

$$-x_1 + 3x_2 \leq 5 \tag{9}$$
$$2x_1 + x_2 \leq 8 \tag{10}$$
$$x_1 \geq 0 \tag{11}$$
$$x_2 \geq 0 \tag{12}$$

and its corresponding set of integer feasible points $X = P \cap \mathbb{Z}^2$.

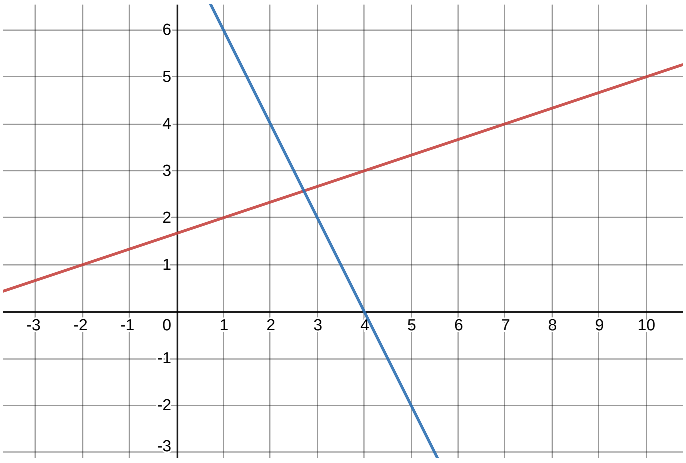


Figure 3: https://www.desmos.com/calculator/exbrptd2ig

(a) Which of the following inequalities are **valid** for X? Which are **redundant** in the description of $P \cap B_i$?

i. $B_1 : x_1 \leq 5$

ii. $B_2 : x_1 + x_2 \geq 1$

iii. $B_3 : x_2 \leq 2$

(b) Use the Chvátal-Gomory procedure with the following constraint weight vectors

i. $\mathbf{u}_1 = (\frac{1}{3}, 0)$

ii. $\mathbf{u}_2 = (\frac{1}{4}, \frac{1}{4})$

to derive two new valid inequalities C_1 and C_2 for X. State which of the newly derived valid inequalities (if any) are facet-defining for $\text{conv}(X)$.

(c) Is the resulting polyhedron $P \cap C_1 \cap C_2$ equal to $\text{conv}(X)$? If not, state the missing inequalities.

(d) Solve the linear program

$$\max\ x_1 + x_2 \tag{13}$$
$$\text{s.t. } \mathbf{x} \in P \tag{14}$$

Does the optimal solution change if you consider the feasible set $P \cap C_1 \cap C_2$ instead?

6. *(6 points) Cover Inequalities*

Consider the knapsack set

$$X = \{\{0,1\}^5 : 9x_1 + 6x_2 + 5x_3 + 4x_4 + x_5 \leq 12\}$$

(a) State a minimum cover C with $|C| \geq 3$ and its corresponding valid cover inequality.

(b) State the **extended** cover inequality for C.

(c) **Lift** the cover inequality for C to obtain a strong valid inequality, i.e., compute all lifting coefficients.

(d) State **all** optimal solutions for the following ILP

$$\max\ x_1 + x_2 + x_3 + x_4 + x_5 \tag{15}$$

$$\text{s.t. } \mathbf{x} \in X \tag{16}$$