VU Introduction to Quantum Computing SS 2024
Exercise Sheet 1

Exercise 1: Let $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$ be two complex numbers. Show:

$$\text{(i)} \quad z_1 z_2 = r_1 r_2(\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)).$$

$$\text{(ii)} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\phi_1 - \phi_2) + i \sin(\phi_1 - \phi_2)).$$

Moreover, for $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, show that

$$\text{(iii)} \quad \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}.$$

Exercise 2: Let V be a vector space with inner product $\langle \cdot, \cdot \rangle$. Show that the function $\|v\| := \sqrt{\langle v, v \rangle}$ is indeed a norm.

Hint: For property (N_3), use the *Cauchy-Schwarz inequality*: $|\langle v, w \rangle| \leq \|v\| \|w\|$, for each $v, w \in V$.

Exercise 3: Let V and $\| \cdot \|$ be as in Exercise 2. Show that $\| \cdot \|$ satisfies the *parallelogram law*:

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

Exercise 4: Show:

(i) The eigenvalues of a self-adjoined operator are always real.

(ii) Projection operators have only 0 and 1 as eigenvalues.

Exercise 5: Let T be a unitary operator. Show:

(i) T^{-1} is unitary.

(ii) $(T^*)^* = T$.

Exercise 6: Let A be an $n \times n$ matrix and B an $m \times m$ matrix. Show that, if both A and B are unitary, then so is $A \otimes B$.

Hint: Note that $(A \otimes B)^* = A^* \otimes B^*$, and, for matrices R, S, T, U, it holds that $(R \otimes S)(T \otimes U) = (RT \otimes SU)$, providing the matrices are such that the multiplications are defined.

Exercise 7: Show that two eigenvectors corresponding to distinct eigenvalues of a self-adjoined operator always are orthogonal to each other.

Exercise 8: Show that the expectation value $\langle M \rangle$ of an observable M is a real number.

Exercise 9: Show that the Pauli matrices

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Hermitian, unitary, and that they square to the identity.

Exercise 10: Two matrices A and B are said to *commute* if the commutator

$$[A, B] := AB - BA = 0.$$

Analogously, two matrices A and B are said to *anticommute* if the *anticommutator*

$$\{A, B\} := AB + BA = 0,$$

i.e., if $AB = -BA$.

Show that any pair of Pauli matrices (cf. Exercise 9) either commute or anticommute.