The Matching Problem

Example ***(The Matching Problem)***

Suppose that each of *N* men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat.

(a) What is the probability that none of the men selects his own hat?

(b) What is the probability that exactly *k* of the men select their own hats?

Solution:

(a)

We first calculate the complementary probability of at least one man's selecting his own hat. Let us denote by $E_i, i = 1, 2, \ldots, N$ the event that the *i*th man selects his own hat. Now, $P\left(\bigcup_{i=1}^{N} E_i\right)$, the probability that at least one of the men selects his own hat, is given by

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots$$
$$+(-1)^{n+1} \sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n})$$
$$+ \cdots + (-1)^{N+1} P(E_1E_2\cdots E_N)$$

If we regard the outcome of this experiment as a vector of *N* numbers, where the *i*th element is the number of the hat drawn by the *i*th man, then there are *N*! possible outcomes. [The outcome $(1, 2, 3, \ldots, N)$ means, for example, that each man selects his own hat.] Furthermore, $E_{i_1}E_{i_2}\ldots E_{i_n}$, the event that each of the *n* men $i_1, i_2, \ldots, i_n$ selects his own hat, can select any of *N*-*n* hats, the second can then select any of *N*-(*n*+1) hats, and so on. Hence, assuming that all *N*! possible outcomes are equally likely, we see that

$$P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{(N-n)!}{N!}$$

Also, as there are $\binom{N}{n}$ terms in $\sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n})$, we see that

$$\sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{N!(N-n)!}{(N-n)!n!N!} = \frac{1}{n!}$$

and thus

$$P\left(\bigcup_{i=1}^{N} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{N+1}\frac{1}{N!}$$

Hence the probability that none of the men selects his own hat is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^N}{N!}$$

which for *N* large is approximately equal to $e^{-1} \approx .36788$. In other words, for *N* large, the probability that none of the men selects his own hat is approximately .37.

(b)

To obtain the probability that exactly *k* of the *N* men select their own hats, we first fix attention on a particular set of *k* men. The number of ways in which these and only these *k* men can select their own hats is equal to the number of ways in which the other *N-k* men can select among their hats in such a way that none of them selects his own hat. But, as

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}$$

is the probability that not one of *N-k* men, selecting among their hats, selects his own, it follows that the number of ways in which the set of men selecting their own hats corresponds to the set of *k* men under consideration is

$$(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right]$$

Hence, as there are $\binom{N}{k}$ possible selections of a group of *k* men, it follows that there are

$$\binom{N}{k}(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right]$$

ways in which exactly *k* of the men select their own hats. The desired probability is thus

$$\frac{\binom{N}{k}(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right]}{N!} = \frac{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}}{k!}$$

which for *N* large is approximately $e^{-1}/k!$. ∎

Source: http://libai.math.ncu.edu.tw/webclass/statistics/probability/notes/ch2_sec5_p2/