1.) a.) $z = f(x,y) = x^2 - y^2$

$$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$$

es gilt: $D = \mathbb{R}^2$, $f(D) = \mathbb{R}$

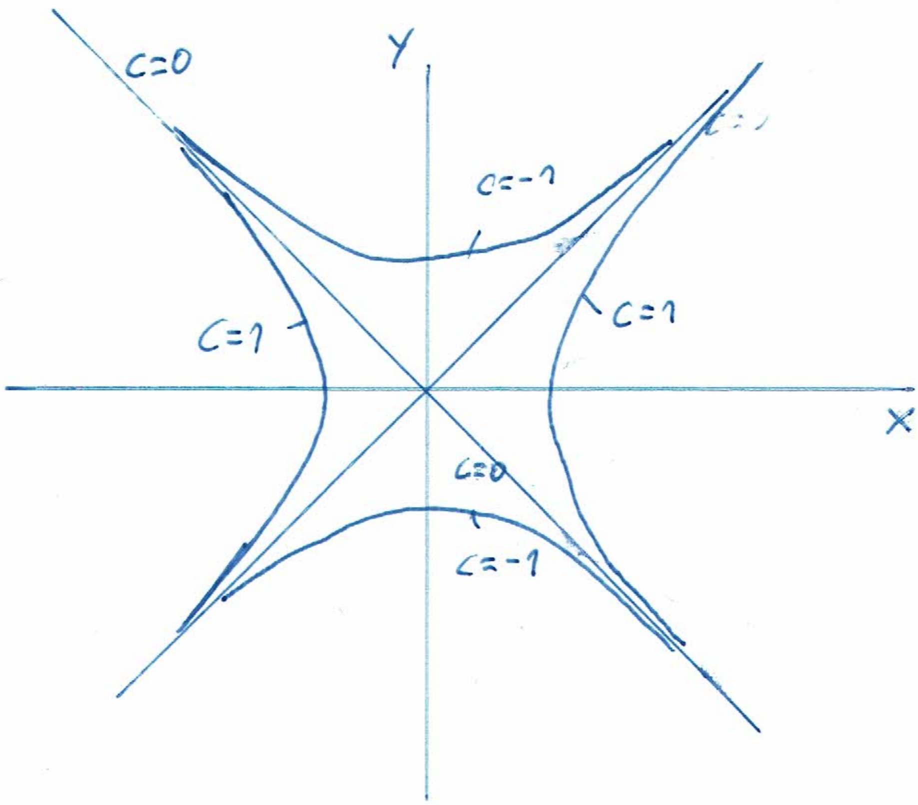
Höhenlinien: $x^2 - y^2 = c$: Hyperbeln

$c > 0$:

$c < 0$:

$c = 0$:
(entartet, 2 Gerade)

Skizze:



1.) b.) $z = f(x,y) = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$

$$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$$

es gilt: $D = \{(x,y) : (\frac{x}{2})^2 + (\frac{y}{3})^2 \leq 1\}$, $f(D) = [0,1]$

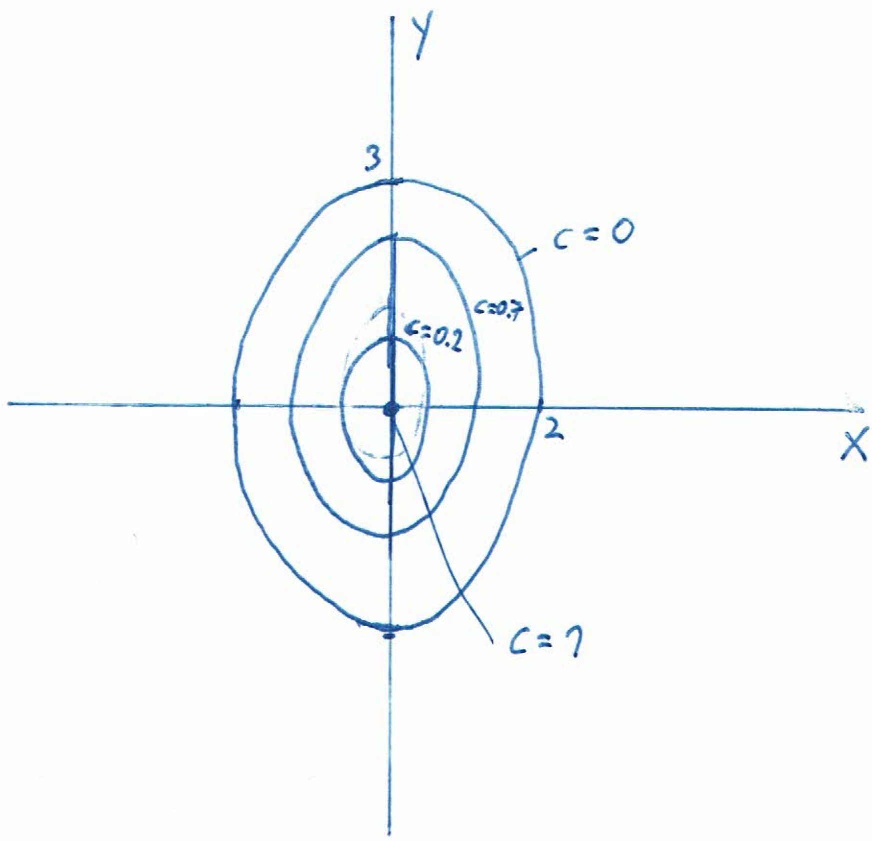
Rand von D: $\partial D : (\frac{x}{2})^2 + (\frac{y}{3})^2 = 1$, Ellipse

Höhenlinien: $\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} = c \in [0,1]$

$$\Rightarrow 1 - \frac{x^2}{4} - \frac{y^2}{9} = c^2$$

$$\Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 - c^2 \quad \text{Ellipsen}$$

Skizze:



8.) a.) $f(x,y) = \sqrt{1-x^2-y^2} = (1-x^2-y^2)^{-\frac{1}{2}}$

$$f_x = \frac{-2x}{2\cdot\sqrt{1-x^2-y^2}} = -\frac{x}{\sqrt{1-x^2-y^2}}$$

$$f_y = -\frac{y}{\sqrt{1-x^2-y^2}}$$

$$(x_0, y_0) = (0.2, 0.3)$$

Tangentialebene τ: $z = f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$

$$\Rightarrow z = 0.9327... - 0.2144...\cdot(x-0.2) - 0.3216...\cdot(y-0.3)$$

$$z = 1.017... - 0.2144...\cdot x - 0.3216...\cdot y$$

b.) $f(x,y) = x^2\cdot\sin y + \cos(x+2y)$

$$f_x = 2x\cdot\sin y - \sin(x+2y)$$

$$f_y = x^2\cdot\cos y - 2\cdot\sin(x+2y)$$

$$f_{xx} = 2\cdot\sin y - \cos(x+2y)$$

$$f_{xy} = 2x\cdot\cos y - 2\cdot\cos(x+2y) = f_{yx}$$

$$f_{yy} = -x^2\cdot\sin y - 4\cdot\cos(x+2y)$$

24.) $y = y(x)$ implizit def. via $x^3 - 3xy + y^3 - 1 = 0$

a.) implizit differenzieren, $y = y(x)$, $\frac{d}{dx}$:

$$3x^2 - 3y - 3xy' + 3y^2 \cdot y' = 0$$

$$\Leftrightarrow \quad x^2 - y = (x - y^2) \cdot y'$$

$$\Leftrightarrow \quad y' = \frac{x^2 - y}{x - y}$$

alternativ: $F(x,y) = x^3 - 3xy + y^3 - 1 = 0$

$$F_x = 3x^2 - 3y, \quad F_y = -3x + 3y^2$$

$$y' = -\frac{F_x}{F_y} = -\frac{3x^2 - 3y}{-3x + 3y^2} = \frac{x^2 - y}{x - y^2}$$

b.) $x = 1$: $1 - 3y + y^3 - 1 = 0$

$$\Updownarrow$$

$$-3y + y^3 = 0$$

$$\Updownarrow$$

$$y \cdot (y^2 - 3) = 0 \Leftrightarrow y \cdot (y - \sqrt{3}) \cdot (y + \sqrt{3}) = 0$$

$$\Rightarrow y_1 = 0, \; y_2 = \sqrt{3}, \; y_3 = -\sqrt{3}$$

$$\Rightarrow P_1(1, 0), \quad P_2(1, \sqrt{3}), \quad P_3(1, -\sqrt{3})$$

$P_2(1, 1.7320..)$ " $P_3(1, -1.7320...)$ "

24., c.) $F_y = 0$; $-x + y^2 = 0 \Leftrightarrow x = y^2$

$\Rightarrow y^6 - 3y^3 + y^3 - 1 = 0$

$y^6 - 2y^3 - 1 = 0$, setze $z := y^3$

$\Rightarrow z^2 - 2z - 1 = 0$

$\Rightarrow z_{1,2} = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2}$

$\Rightarrow y_1 = \sqrt[3]{1+\sqrt{2}}$, $y_2 = -\sqrt[3]{\sqrt{2}-1}$

$x_1 = (1+\sqrt{2})^{\frac{2}{3}}$, $x_2 = (\sqrt{2}-1)^{\frac{2}{3}}$

$\Rightarrow Q_1(1.7996..., 1.3415...)$, $Q_2(0.5556..., -0.7454...)$

27.) $F(x,y) = f(g(x,y), h(x,y))$ mit $f(u,v) = u^2 + v^2$,

$u = g(x,y) = \cos x + \sin y$, $v = h(x,y) = x + y + 1$

ges. $F_y(0,0)$ mit Kettenregeln

$$F_y = "\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}"$$

$$F_y(x,y) = \frac{\partial f}{\partial u}(g(x,y), h(x,y)) \cdot \frac{\partial u}{\partial y}(x,y) + \frac{\partial f}{\partial v}(g(x,y), h(x,y)) \cdot \frac{\partial v}{\partial y}(x,y)$$

$$\frac{\partial f}{\partial u} = 2u, \quad \frac{\partial f}{\partial v} = 2v, \quad \frac{\partial u}{\partial y} = g_y = \cos y, \quad \frac{\partial v}{\partial y} = h_y = 1$$

$$\Rightarrow F_y(x,y) = 2u \cdot \cos y + 2v \cdot 1 = 2 \cdot (\cos x + \sin y) \cdot \cos y + 2 \cdot (x + y + 1)$$

$$\Rightarrow F_y(0,0) = 2 \cdot 1 \cdot 1 + 2 \cdot 1 = 4$$

29.) $q(\vec{x}) = q(x,y) = 4x^2 + 2bxy + 25y^2, \; b \in \mathbb{R},$

quadratische Form

$\Rightarrow \quad A = \begin{pmatrix} 4 & b \\ b & 25 \end{pmatrix}$

$\Rightarrow \quad q(\vec{x}) = \vec{x}^T \cdot A \cdot \vec{x} = (x \; y) \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$\downarrow$

verw. Spaltenvektor $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

Hauptminorenkrit. anwenden:

$\det a_{11} = a_1 = 4 > 0$ ✓

$\det A = 4 \cdot 25 - b^2 = 100 - b^2 > 0$

$\Leftrightarrow \quad 100 > b^2 \quad \Leftrightarrow \quad |b| < 10$

$\Rightarrow$ quadratische Form positiv definit für $|b| < 10$

110.) $$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \sin(t+\frac{\pi}{2}) \\ \sin(3t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin(3t) \end{pmatrix}$$

$$\frac{dy}{dx} = \frac{dy(t)}{dt} \cdot \frac{dt}{dx(t)} = \frac{\dot{y}(t)}{\dot{x}(t)}$$

$$\dot{x} = -\sin t, \quad \dot{y} = 3\cdot\cos(3t)$$

$$\Rightarrow y'(x) = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3\cdot\cos(3t)}{-\sin t} = -3\cdot\frac{\cos(3t)}{\sin t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right)\cdot\frac{dt}{dx} =$$

$$= \frac{\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)}{\dot{x}} = \frac{\frac{\ddot{y}\cdot\dot{x}-\dot{y}\cdot\ddot{x}}{\dot{x}^2}}{\dot{x}} = \frac{\ddot{y}\cdot\dot{x}-\dot{y}\cdot\ddot{x}}{\dot{x}^3}$$

$$\ddot{x} = -\cos t, \quad \ddot{y} = -9\cdot\sin(3t)$$

$$\Rightarrow y''(x) = \frac{d^2y}{dx^2} = \frac{-9\cdot\sin(3t)\cdot(-\sin t) - 3\cdot\cos(3t)\cdot(-\cos t)}{(-\sin t)^3} =$$

$$= -\frac{9\cdot\sin t\cdot\sin(3t) + 3\cdot\cos t\cdot\cos(3t)}{\sin^3 t}$$

horizontale Tangenten:

$\dot{y} = 0$: $3\cdot\cos(3t) = 0 \Leftrightarrow \cos(3t) = 0$

$\cos u = 0 \Leftrightarrow u = \frac{\pi}{2} + \pi\cdot k, \ k\in\mathbb{Z}$

$\cos(3t) = 0 \Leftrightarrow t = \frac{\pi}{6} + \frac{\pi\cdot k}{3}, \ k\in\mathbb{Z}$

$t\in[0,2\pi)$: $t_1 = \frac{\pi}{6}, \ t_2 = \frac{3\pi}{6}, \ t_3 = \frac{5\pi}{6}, \ t_4 = \frac{7\pi}{6}, \ t_5 = \frac{9\pi}{6}, \ t_6 = \frac{11\pi}{6}$

ad 110.,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin(3t) \end{pmatrix}$$

$\Rightarrow P_1(0.866\ldots, 1),\quad P_2(0, -1),\quad P_3(-0.866\ldots, 1),$

$P_4(-0.866\ldots, -1),\quad P_5(0, 1),\quad P_6(0.866\ldots, -1)$

vertikale Tangenten:

$\dot{x} = 0: \quad -\sin t = 0 \Leftrightarrow \sin t = 0$

$t \in [0, 2\pi): \quad t_1 = 0, \quad t_2 = \pi$

$\Rightarrow Q_1(1, 0),\quad Q_2(-1, 0)$

Wendepunkte:

$y''(x) = 0 \quad : \quad \ddot{y} \cdot \dot{x} - \dot{y} \cdot \ddot{x} = 0$

$\Leftrightarrow \quad -9 \cdot \sin(3t) \cdot (-\sin t) - 3 \cdot \cos(3t) \cdot (-\cos t) = 0$

$\Leftrightarrow \quad 3 \cdot \sin t \cdot \sin(3t) = -\cos t \cdot \cos(3t)$

$\Leftrightarrow \quad 3 \cdot \tan t \cdot \tan(3t) = -1$: Lösen mittels Computeralgebrasystem

oder: Summensätze: $\sin(3t) = 3 \cdot \sin t \cdot \cos^2 t - \sin^3 t$

$\cos(3t) = \cos^3 t - 3 \cos t \cdot \sin^2 t$

$$\Rightarrow \tan(3t) = \frac{\sin(3t)}{\cos(3t)} = \frac{3 \cdot \tan t - \tan^3 t}{1 - 3 \cdot \tan^2 t}$$

$\Rightarrow \quad 3 \cdot \tan t \cdot \tan(3t) = -1$

$\Rightarrow \quad 3 \cdot \tan t \cdot (3 \cdot \tan t - \tan^3 t) = -(1 - 3 \tan^2 t)$

$\Rightarrow \quad 3 \tan^4 t - 6 \tan^2 t - 1 = 0$

ad 110.)

$\tan^2 t = u$

$\Rightarrow \quad 3u^2 - 6u - 1 = 0$

$$u_{1,2} = \frac{6 \pm \sqrt{48}}{6} = 1 \pm \frac{2\cdot\sqrt{3}}{3} = \begin{cases} 2.1547\ldots \\ -0.1547\ldots \end{cases}$$

$\Rightarrow \quad \tan^2 t = 2.1547\ldots$

$\Rightarrow \quad \tan t = \pm 1.4678\ldots$

$t \in [0, 2\pi):\quad t_1 = 0.9727\ldots,\quad t_2 = 1.6737\ldots,\quad t_3 = 4.1143\ldots,\quad t_4 = 4.8152\ldots$

$\Rightarrow \quad W_1(0.5630\ldots, 0.2214\ldots),\ W_2(-0.5630\ldots, 0.2214\ldots),$

$W_3(-0.5630\ldots, -0.2214\ldots),\ W_4(0.5630\ldots, -0.2214\ldots)$

Od. bsp. 110.)

> $solve(3\cdot\sin(t)\cdot\sin(3\cdot t) = -\cos(t)\cdot\cos(3\cdot t), t);$

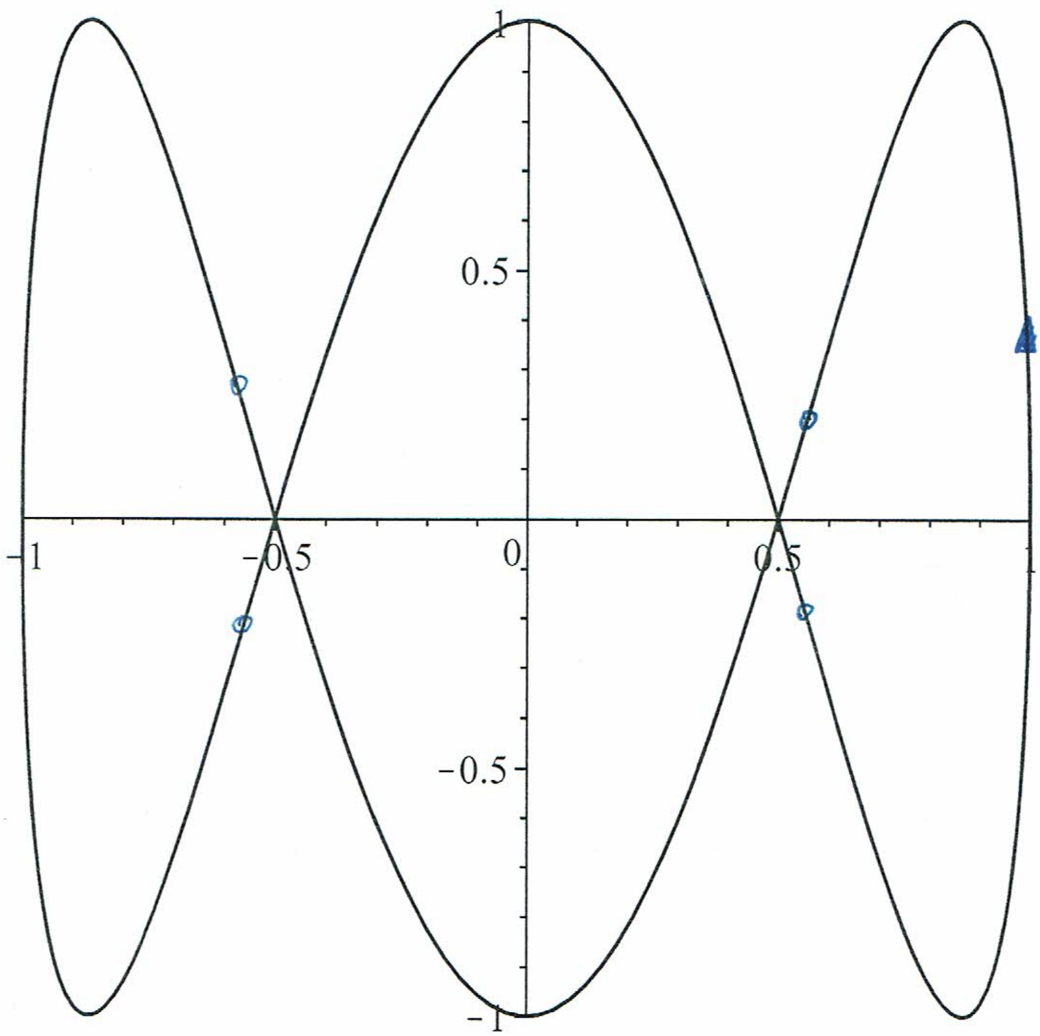
$$\arctan\left(\sqrt{1-\sqrt{3}}, \sqrt{3+\sqrt{3}}\right), \arctan\left(-\frac{1}{2}\sqrt{1-\sqrt{3}}, \frac{1}{2}\sqrt{3+\sqrt{3}}\right),$$
$$\arctan\left(\frac{1}{2}\sqrt{1-\sqrt{3}}, -\frac{1}{2}\sqrt{3+\sqrt{3}}\right), \arctan\left(-\frac{1}{2}\sqrt{1-\sqrt{3}}, -\frac{1}{2}\sqrt{3+\sqrt{3}}\right),$$
$$\arctan\left(\frac{\sqrt{1+\sqrt{3}}}{\sqrt{3-\sqrt{3}}}\right), -\arctan\left(\frac{\sqrt{1+\sqrt{3}}}{\sqrt{3-\sqrt{3}}}\right), -\arctan\left(\frac{\sqrt{1+\sqrt{3}}}{\sqrt{3-\sqrt{3}}}\right) + \pi,$$
$$\arctan\left(\frac{\sqrt{1+\sqrt{3}}}{\sqrt{3-\sqrt{3}}}\right) - \pi \qquad (1)$$

> $evalf(\%);$

$$0.4157214729\,I, -0.4157214727\,I, 3.141592654 - 0.4157214727\,I, 3.141592654 + 0.4157214727\,I, 0.9727653801, -0.9727653801, 2.168827274, -2.168827274 \qquad (2)$$

> ? $plot$

> $plot([\cos(t), \sin(3\cdot t), t = 0..2\cdot Pi]);$



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