48.) $f(x,y) = \cos(x+y) + \sin x + \sin y \,, \quad 0 \leq x,y \leq \frac{\pi}{2}$

$$D = [0,\tfrac{\pi}{2}] \times [0,\tfrac{\pi}{2}]$$

• rel. Extrema im Inneren von D, $\mathring{D}$:

$$\left.\begin{array}{ll} \text{I}: & f_x = -\sin(x+y) + \cos x = 0 \\ \text{II}: & f_y = -\sin(x+y) + \cos y = 0 \end{array}\right\} \text{ stat. Punkte}$$

$\text{I} - \text{II}: \Rightarrow \quad \cos x - \cos y = 0 \Rightarrow \cos x = \cos y$

da $\cos x$ auf $[0,\frac{\pi}{2}]$ injektiv

$\Rightarrow \quad x = y$

$\text{I}: \quad \cos x - \sin(2x) = 0$

$\Rightarrow \cos x = \sin(2x) = 2 \cdot \sin x \cdot \cos x$

$\Rightarrow \cos x \cdot (1 - 2 \cdot \sin x) = 0$

Fall: $\cos x = 0 \Rightarrow x = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2}$; NICHT in $\mathring{D}$
Rand später betrachtet

Fall: $1 - 2 \cdot \sin x = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} \Rightarrow y = \frac{\pi}{6}$

$\Rightarrow$ einziger stat. Punkt in $\mathring{D}$: $P_1(\frac{\pi}{6}, \frac{\pi}{6})$

ad 48.)

Hesse-Matrix betrachten: $H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$

$f_{xx} = -\cos(x+y) - \sin x$

$f_{xy} = f_{yx} = -\cos(x+y) \Rightarrow H_f(x,y) = \begin{pmatrix} -\cos(x+y)-\sin x & -\cos(x+y) \\ -\cos(x+y) & -\cos(x+y)-\sin y \end{pmatrix}$

$f_{yy} = -\cos(x+y) - \sin y$

$\Rightarrow H_f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}$

$f_{xx} < 0$, $\det H_f = 1 - \frac{1}{4} > 0$

$\Rightarrow H_f$ neg. definit $\Rightarrow$ f besitzt an Stelle $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$ ein rel. Maximum

Funktionswert $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$

- Rand ∂D von D betrachten:

  - $y = 0$, $x \in [0, \frac{\pi}{2}]$:

    $f_1(x) = f(x,0) = \cos x + \sin x$

    $f_1'(x) = -\sin x + \cos x = 0 \Rightarrow \sin x = \cos x$

    $\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

    $f_1''(x) = -\cos x - \sin x < 0$ für $x \in [0, \frac{\pi}{2}]$ (konkav)

    $f_1\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \sqrt{2} \approx 1.41.. \Rightarrow \max f_1$

    $f_1(0) = f_1\left(\frac{\pi}{2}\right) = 1 \Rightarrow \min f_1$

  - $y = \frac{\pi}{2}$, $x \in [0, \frac{\pi}{2}]$:

    $f_2(x) = f\left(x, \frac{\pi}{2}\right) = \underbrace{\cos\left(x+\frac{\pi}{2}\right)}_{-\sin x} + \sin x + \underbrace{\sin\frac{\pi}{2}}_{1} = 1$

    $\min f_2 = \max f_2 = 1$

ad 48.)

Symmetrie: $f(0,y) = f(y,0) = f_1(y), \quad y \in [0, \frac{\pi}{2}]$

$$f(\tfrac{\pi}{2}, y) = f(y, \tfrac{\pi}{2}) = f_2(y), \quad y \in [0, \tfrac{\pi}{2}]$$

$$\Rightarrow \quad \max_{(x,y)\in D} f = \max\left\{ \max_{(x,y)\in \mathring{D}} f, \max_{x\in[0,\frac{\pi}{2}]} f_1, \max_{x\in[0,\frac{\pi}{2}]} f_2 \right\} = \frac{3}{2}$$

$$\min_{(x,y)\in D} f = \min\left\{ \min_{x\in[0,\frac{\pi}{2}]} f_1, \min_{x\in[0,\frac{\pi}{2}]} f_2 \right\} = 1$$

51.) $f(x,y,z) = 2x^2 - 3xz^2 + y^3 + 3z^2 - 3y + 3$

$$\left.\begin{aligned} \text{I}: \; f_x &= 4x - 3z^2 = 0 \\ \text{II}: \; f_y &= 3y^2 - 3 = 3\cdot(y^2-1) = 0 \\ \text{III}: \; f_z &= -6xz + 6z = 6z\cdot(1-x) = 0 \end{aligned}\right\} \text{stat. Punkte ermitteln}$$

II: $y^2 = 1 \Rightarrow y = \pm 1$

III: • entweder $z = 0$

$\Rightarrow$ I: $x = 0$

$\Rightarrow P_1(0,1,0)$, $P_2(0,-1,0)$

• oder $1 - x = 0 \Rightarrow x = 1$

$\Rightarrow$ I: $z^2 = \frac{4x}{3} = \frac{4}{3}$

$\Rightarrow z = \pm \frac{2}{\sqrt{3}}$

$\Rightarrow P_3(1,1,\frac{2}{\sqrt{3}})$, $P_4(1,1,-\frac{2}{\sqrt{3}})$,

$P_5(1,-1,\frac{2}{\sqrt{3}})$, $P_6(1,-1,-\frac{2}{\sqrt{3}})$

$\Rightarrow$ 6 stat. Punkte

Hesse-Matrix:

$f_{xx} = 4$, $f_{xy} = 0$, $f_{xz} = -6z$, $f_{yy} = 6y$, $f_{yz} = 0$, $f_{zz} = 6\cdot(1-x)$

$$H_f(x,y,z) = \begin{pmatrix} 4 & 0 & -6z \\ 0 & 6y & 0 \\ -6z & 0 & 6\cdot(1-x) \end{pmatrix}$$

ad 51,

$P_1(0,1,0)$: $H_f = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

H_f positiv definit (Hauptminorenkrit) oder Eigenwerte betr.

$\Rightarrow$ P_1 rel. ~~Maximum~~ Min.

$P_2(0,-1,0)$: $H_f = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

H_f indefinit (Hauptminorenkrit oder EW, 4, −6)

$\Rightarrow$ P_2 Sattelpunkt

$P_{3,4}(1,1,\pm\frac{2}{\sqrt{3}})$: $H_f = \begin{pmatrix} 4 & 0 & \pm\frac{12}{\sqrt{3}} \\ 0 & 6 & 0 \\ \pm\frac{12}{\sqrt{3}} & 0 & 0 \end{pmatrix}$

H_f indefinit: Hauptminorenkrit:

$M_1 > 0, M_2 > 0, M_3 < 0$

$\Rightarrow$ P_3, P_4 Sattelpunkte

$P_{5,6}(1,-1,\pm\frac{2}{\sqrt{3}})$: $H_f = \begin{pmatrix} 4 & 0 & \pm\frac{12}{\sqrt{3}} \\ 0 & -6 & 0 \\ \pm\frac{12}{\sqrt{3}} & 0 & 0 \end{pmatrix}$

H_f indefinit: Hauptminorenkrit:

$M_1 = 4 > 0$, $M_2 = -24 < 0$

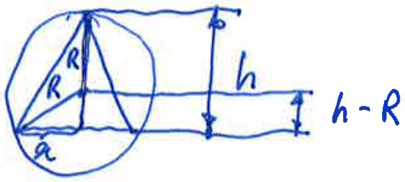
$M_3 > 0$

$\Rightarrow$ P_5, P_6 Sattelpunkte

63.) Kugel, Drehkegel mit max. Volumen einschreiben

Kugel: Radius R

Querschnitt:



$$V_\Delta = \frac{r^2 \cdot \pi \cdot h}{3} \to \max.$$

NB: $(h-R)^2 + r^2 = R^2$

$$\Rightarrow (h-R)^2 + r^2 - R^2 = 0$$

Lagrange-Fkt.:

$$F(r, h, \lambda) = \frac{r^2 \cdot \pi \cdot h}{3} - \lambda \cdot \left((h-R)^2 + r^2 - R^2 \right)$$

I: $F_r = \dfrac{2r \cdot \pi \cdot h}{3} - \lambda \cdot 2r = 0$

II: $F_h = \dfrac{r^2 \cdot \pi}{3} - 2\lambda \cdot (h-R) = 0$

III: $F_\lambda = -\left((h-R)^2 + r^2 - R^2 \right) = 0$

I: $2r \cdot \left(\dfrac{\pi \cdot h}{3} - \lambda \right) = 0$

$(r = 0 : \min) \Rightarrow \dfrac{\pi \cdot h}{3} - \lambda = 0$

$$\Rightarrow \lambda = \frac{\pi \cdot h}{3}$$

ad 63.)

$$\text{II} \Rightarrow \quad \frac{r^2\cdot\pi}{3} - \frac{2\cdot\pi\cdot h}{3}\cdot(h-R) = 0$$

$$\Rightarrow \quad r^2 - 2h\cdot(h-R) = 0$$

$$\Rightarrow \quad r^2 = 2h\cdot(h-R)$$

$$\text{III} \Rightarrow \quad (h-R)^2 + r^2 = R^2$$

$$\Rightarrow (h-R)^2 + 2h\cdot(h-R) = R^2$$

$$\Rightarrow h^2 - 2hR + \cancel{R^2} + 2h^2 - 2hR = \cancel{R^2}$$

$$\Rightarrow \quad 3h^2 - 4hR = 0$$

$$h\cdot(3h-4R) = 0$$

$$(h=0 \Rightarrow \min) \quad \Rightarrow \quad 3h-4R = 0$$

$$\Rightarrow \quad h = \frac{4R}{3},$$

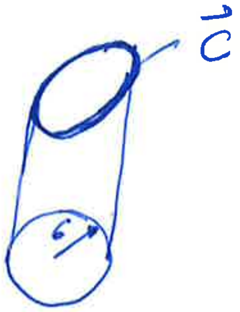
$$\Rightarrow r^2 = 2h\cdot(h-R) = \frac{8R}{3}\cdot\frac{R}{3} = \frac{8R^2}{9}$$

$$r = \frac{2\cdot\sqrt{2}\cdot R}{3}$$

$$\Rightarrow \quad V_A = \frac{r^2\cdot\pi\cdot h}{3} = \frac{32}{81}R^3\cdot\pi$$

max. Volumen für obige r, h.

67.)



$NB_1: \quad z = x + 2y + 55$

$\Rightarrow \quad z - x - 2y - 55 = 0$

$NB_2: \quad x^2 + y^2 = 6^2 = 36$

$\Rightarrow \quad x^2 + y^2 - 36 = 0$

beschreiben Kurve $\vec{c}$ (die Ellipse)

Zielfunktion: $z \to \max.$

Lagrange-Fkt.:

$$F(x, y, z, \lambda_1, \lambda_2) = z - \lambda_1 \cdot (z - x - 2y - 55) - \lambda_2 \cdot (x^2 + y^2 - 36)$$

$\mathrm{I}: \; F_x = \lambda_1 - 2\lambda_2 x = 0$

$\mathrm{II}: \; F_y = 2\lambda_1 - 2\lambda_2 y = 0$

$\mathrm{III}: \; F_z = 1 - \lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = 1$

$\mathrm{IV}: \; F_{\lambda_1} = -(z - x - 2y - 55) = 0$

$\mathrm{V}: \; F_{\lambda_2} = -(x^2 + y^2 - 36) = 0$

$\mathrm{III}: \; \lambda_1 = 1$

aus I & II: $\quad 1 - 2\lambda_2 x = 0$

$\qquad 2 - 2\lambda_2 y = 0 \Rightarrow 1 - \lambda_2 y = 0$

$\Leftrightarrow \quad \lambda_2 \cdot (y - 2x) = 0$

$\overset{\lambda_2 \neq 0}{\Rightarrow} \quad y - 2x = 0 \quad \Rightarrow \quad y = 2x$

sonst $F_x \neq 0$

ad 67.)

$y = 2x$ einsetzen in ∇:

$$x^2 + (2x)^2 = 36$$

$$\Rightarrow \quad 5x^2 = 36 \Rightarrow x^2 = \frac{36}{5} \Rightarrow x_{1,2} = \pm \frac{6}{\sqrt{5}}$$

$$y = 2x \Rightarrow \quad y_{1,2} = \pm \frac{12}{\sqrt{5}}$$

$P_1\left(\frac{6}{\sqrt{5}}, \frac{12}{\sqrt{5}}\right)$, $P_2\left(-\frac{6}{\sqrt{5}}, -\frac{12}{\sqrt{5}}\right)$ stat. Punkte

offensichtlich liefert P_1 das Maximum:

∇ kompakt, Max. existiert und befindet sich unter stat. Punkten

$$\Rightarrow \quad z_{max} = \frac{30}{\sqrt{5}} + 55 = 6 \cdot \sqrt{5} + 55 \approx 68.42 \, m$$

72.1 $f(x,y,z) = x + y + z^2$

$NB: \quad x^2 - y^2 + z^2 = 1 \;, \quad x + y = 1$

Lagrange - Fkt.:

$$F(x,y,z,\lambda_1,\lambda_2) = x + y + z^2 - \lambda_1 \cdot (x^2 - y^2 + z^2 - 1) - \lambda_2 \cdot (x + y - 1)$$

$I: \quad F_x = 1 - 2x\lambda_1 - \lambda_2 = 0$

$II: \quad F_y = 1 + 2y\lambda_1 - \lambda_2 = 0$

$III: \quad F_z = 2z - 2z\lambda_1 = 2z \cdot (1 - \lambda_1) = 0$

$IV: \quad F_{\lambda_1} = -(x^2 - y^2 + z^2 - 1) = 0$

$V: \quad F_{\lambda_2} = -(x + y - 1) = 0$

$I \& II: \quad II - I = 2\lambda_1 \cdot (x + y) = 0$

Fall: $x + y = 0$: $\lightning$ zu V, da $x + y = 1$

$\Rightarrow \lambda_1 = 0$

$I \Rightarrow \lambda_2 = 1$

$III \Rightarrow z = 0$

$IV \Rightarrow \underbrace{x^2 - y^2}_{(x-y)\cdot(x+y)} = 1$

$(V: x + y = 1) \Rightarrow x - y = 1$

$\Rightarrow \begin{matrix} x - y = 1 \\ x + y = 1 \end{matrix} \Rightarrow x = 1, \; y = 0$

$\Rightarrow$ einzige stat. Punkt $P(1,0,0)$

76.) $I = \iint_B e^{2x} \cdot (y+1)\, dx\, dy$

Rechteckbereich $B = [-2,4] \times [0,3]$

$$\Rightarrow I = \int_{-2}^{4} \int_{0}^{3} e^{2x} \cdot (y+1)\, dy\, dx =$$

$$= \int_{-2}^{4} e^{2x} \cdot \left(\int_{0}^{3} (y+1)\, dy \right) dx =$$

$$= \int_{-2}^{4} e^{2x} \cdot \left(\frac{y^2}{2} + y \Big|_{0}^{3} \right) dx = \int_{-2}^{4} e^{2x} \cdot \left(\frac{9}{2} + 3 \right) \cdot dx =$$

$$= \frac{15}{2} \cdot \int_{-2}^{4} e^{2x}\, dx = \frac{15}{2} \cdot \frac{e^{2x}}{2} \Big|_{-2}^{4} =$$

$$= \frac{15}{4} \cdot \left(e^{8} - e^{-4} \right) \approx 11178.52$$