215) z.z: $\mathcal{L}\{f(at)\}(s) = \frac{1}{a}\mathcal{L}\{f(t)\}(\frac{s}{a})$

$$\int_0^\infty e^{-st} f(t)\, dt$$

$$\int_0^\infty e^{-st} f(at)\, dt \qquad u = at$$

$$\int_0^\infty e^{-st} f(u) \frac{du}{a} \qquad t = \frac{u}{a}$$

$$\frac{1}{a}\cdot\int_0^{\infty\cdot a} e^{-\frac{s}{a}u} f(u)\, du$$

$$\frac{1}{a}\cdot F\left(\frac{s}{a}\right)$$

a) $t \cdot \cos(6t)$

$$\int_0^\infty e^{-st} \cdot t \cdot \cos(6t)\, dt \qquad \mathcal{L}\{t\cdot f(t)\} = -\frac{d}{ds} F(s)$$

$$F(s) = \int_0^\infty e^{-st} \cdot \cos(6t) = \mathcal{L}\{\cos(6t)\} = \frac{s}{s^2+6^2}$$

$$-\frac{d}{ds} F(s) = -\frac{(s^2+6^2) - s\cdot 2s}{(s^2+6^2)^2} = \frac{s^2-6^2}{(s^2+6^2)^2}$$

b) $t^2 \cdot \cos(7t)$

$$\mathcal{L}\{t^2\cdot f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) = \frac{d^2}{ds^2} F(s)$$

$$F(s) = \mathcal{L}\{\cos(7t)\} = \frac{s}{s^2+7}$$

$$\frac{d}{ds} = -\frac{s^2-7^2}{(s^2+7^2)^2} \quad \text{analog zu a)}$$

2.15)

b) $$\frac{d}{ds} \; - \frac{s^2-7^2}{(s^2+7^2)^2} = - \frac{2s\cdot(s^2+7^2)^2-(s^2-7^2)\cdot 2(s^2+7^2)\cdot 2s}{(s^2+7^2)^4}$$

$$= - \frac{2s\left((s^2+7^2)^2-2\cdot(s^2-7^2)\cdot(s^2+7^2)\right)}{(s^2+7^2)^4}$$

$$= - \frac{2s(s^2+7^2)\left((s^2+7^2)-2(s^2-7^2)\right)}{(s^2+7^2)^4}$$

$$= - \frac{2s\left(s^2+7^2-2s^2+2\cdot 7^2\right)}{(s^2+7^2)^3}$$

$$= - \frac{2s(3\cdot 7^2-s^2)}{(s^2+7^2)^3}$$

$$= - \frac{6s\cdot 7^2-2s^3}{(s^2+7^2)^3} = \mathcal{L}\{t^2\cdot\cos(7t)\}$$

222)

a)

$$1 * 2 = \int_0^t 1 \cdot 2 \; d\tau = 2\tau \Big|_0^t = \underline{\underline{2t}}$$ Faltungsprodukt

$$\mathcal{L}\{(1*2)(t)\} = F(s) \cdot G(s) = \frac{1}{s} \cdot \frac{2}{s} = \frac{2}{s^2}$$

$$\int_0^\infty e^{-st} \, 2 \; dt = 2 \cdot \int_0^\infty e^{-st} \; dt = 2 \cdot \mathcal{L}\{1\} = \frac{2}{s}$$

b)

$$e^t * e^{2t} = \int_0^t e^{t-\tau} \cdot e^{2\tau} = \int_0^t e^{t+\tau} \; d\tau = e^{t+\tau}\Big|_0^t = e^{2t} - e^t$$

$$\mathcal{L}\{e^t * e^{2t}\} = F(s) \cdot G(s)$$

$$\int_0^\infty e^{-st} e^t \; dt = \int_0^\infty e^{-st+t} \; dt = \int_0^\infty e^{t(1-s)} \; dt$$

$$= \frac{e^{t(1-s)}}{1-s}\Big|_0^\infty = \lim_{t\to\infty} \frac{e^{t(1-s)}}{1-s} \begin{cases} 0 & s > 1 \text{ konvergiet} \\ \infty & s \leq 1 \end{cases}$$

$$\Rightarrow -\frac{1}{1-s} = \mathcal{L}\{e^t\}$$

$$\mathcal{L}\{e^{2t}\} = \frac{e^{t\cdot(2-s)}}{2-s}\Big|_0^\infty \quad \text{analog zu vorher} \begin{cases} 0 & s \geq 2 \\ \infty & s \leq 2 \end{cases}$$

$$= -\frac{1}{2-s}$$

$$\mathcal{L}\{e^t * e^{2t}\} = \frac{1}{1-s} \cdot \frac{1}{2-s} = \frac{1}{(1-s)(2-s)}$$

Rücktransformation

$$\frac{1}{(1-s)(2-s)} = \frac{1}{s-2} - \frac{1}{s-1} = \mathcal{L}\left\{\frac{1}{s-2} - \frac{1}{s-1}\right\} = e^{2t} - e^t$$

224)

$$x(t) = t^2 + \int_0^t x(y) \cdot \sin(t-y)\, dy$$

Faltung

$$X(s) = \mathcal{L}\{t^2\} + \overbrace{\mathcal{L}\{x(t)\}}^{X(s)} \cdot \mathcal{L}\{\sin(t)\}$$

$$1 = \frac{\mathcal{L}\{t^2\}}{X(s)} + \mathcal{L}\{\sin(t)\}$$

$$\frac{1 - \mathcal{L}\{\sin(t)\}}{\mathcal{L}\{t^2\}} = \frac{1}{X(s)}$$

$= \frac{2}{s^2}$ mit Diff. im Bildbereich

$$\frac{\mathcal{L}\{t^2\}}{1 - \mathcal{L}\{\sin(t)\}} = X(s)$$

$$\frac{2s^2 + 2}{s^5} = X(s)$$

$$\frac{2}{s^5} + \frac{2}{s^3} = X(s)$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^5} + \frac{2}{s^3}\right\} = x(t)$$

$$\mathcal{L}^{-1}\left\{\overbrace{\frac{1}{12} \cdot \frac{24}{s^5}}^{\text{4. Ableitung von } \frac{1}{s}}\right\} + t^2 = x(t)$$

$$\frac{1}{12} \cdot t^4 + t^2 = x(t)$$

228)

a) $\frac{s+3}{s\cdot(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$

$A\cdot(s-1)(s+2) + B\cdot s(s+2) + C\cdot s\cdot(s-1) = s+3$

$s^2\underbrace{(A+B+C)}_{0} + s\underbrace{(A+2B-C)}_{1} \underbrace{-2A}_{3} = s+3$

$-2A = 3$

$A = -1{,}5$

$-1{,}5 + B + C = 0$

$B = 1{,}5 - C$

$-1{,}5 + 3 - 2C - C = 1$

$C = \frac{1}{6}$

$B = 1{,}5 - \frac{1}{6}$

$B = \frac{4}{3}$

$\mathcal{L}^{-1}\left\{-1{,}5\cdot\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{3}\cdot\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{6}\cdot\frac{1}{s+2}\right\}$

$= -1{,}5 + \frac{4}{3}\cdot e^{t} + \frac{1}{6}\cdot e^{-2t}$

228)

b) $\dfrac{s-1}{s^2+2s-8} = \dfrac{s-1}{(x-2)(x+4)} = \dfrac{A}{x-2} + \dfrac{B}{x+4}$

$A\cdot(x+4) + B\cdot(x-2) = s-1$

$x\cdot\underbrace{(A+B)}_{1} + \underbrace{4A-2B}_{-1} = s-1$

$A = 1-B$

$4\cdot(1-B) - 2B = -1$

$B = \frac{5}{6}$

$A = 1-\frac{5}{6} = \frac{1}{6}$

$\mathcal{L}^{-1}\left\{\frac{1}{6}\cdot\frac{1}{x-2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{6}\cdot\frac{1}{x+4}\right\} = \frac{1}{6}\cdot e^{2t} + \frac{5}{6}\cdot e^{-4t}$

228)

c) $\dfrac{3s+7}{s^2-2s+5} = 3\cdot\dfrac{(s-1)}{(s-1)^2+4} + 10\cdot\dfrac{1}{(s-1)^2+4}$

$= 3\cdot\mathcal{L}^{-1}\left\{\dfrac{s-1}{(s-1)^2+2^2}\right\} + 10\cdot\mathcal{L}^{-1}\left\{\dfrac{\frac{1}{2}\cdot 2}{(s-1)^2+2^2}\right\}$

$= 3\cdot e^{t}\cdot\cos(2t) + 5\cdot e^{t}\cdot\sin(2t)$

Verschiebung im Bildbereich

228)

d)

$$\mathcal{L}\left\{\frac{e^{-7s}}{(s+3)^3}\right\}$$ Verschiebung im Zeitbereich

$$e^{-as} \cdot F(s) = \mathcal{L}\{ f(t-a) \cdot u(t-a)\}$$

$$F(s) = \frac{1}{(s+3)^3} \qquad a = 7$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\}$$ Verschiebung im Bildbereich

$$e^{-3t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{2}{s^3}\right\} = \frac{e^{-3t}}{2} \cdot t^2 = f(t)$$

$$\mathcal{L}\left\{f(t-7) \cdot u(t-7)\right\} = \frac{e^{-7s}}{(s+3)^3} \Rightarrow \frac{e^{-3(t-7)}}{2} \cdot (t-7)^2 \cdot u(t-7)$$

$u=$ Heaveside Funktion $\begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

230)

b)

$$y'' - 8y' + 16y = e^{3x} \qquad y(0) = y'(0) = 0$$

$$s^2 \cdot Y(s) - 8s \cdot Y(s) + 16\,Y(s) = \frac{1}{s-3}$$

$$Y(s) \cdot (s^2 - 8s + 16) = \frac{1}{s-3}$$

gemäß Differenziation im Zeitbereich

$$Y(s) = \frac{1}{(s^2 - 8s + 16)(s-3)}$$

$$Y(s) = \frac{1}{(s-4)^2 (s-3)} = \frac{A}{(s-4)} + \frac{B}{(s-4)^2} + \frac{C}{(s-3)}$$

Partialbruchzerlegung

$$1 = A \cdot (s-4)(s-3) + B(s-3) + C \cdot (s-4)^2$$

$$1 = s^2 \underbrace{(A + C)}_{0} + s\underbrace{(-7A + B - 8C)}_{0} + \underbrace{(12A - 3B + 16C)}_{1}$$

$$A + C = 0$$

$$A = -C$$

$$7C + B - 8C = 0$$

$$B = C$$

$$-12C - 3C + 16C = 1$$

$$C = 1 \Rightarrow B = 1 \Rightarrow A = -1$$

$$\mathcal{L}^{-1}\left\{-\frac{1}{(s-4)} + \frac{1}{(s-4)^2} + \frac{1}{(s-3)}\right\} = -e^{4t} + e^{4t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + e^{3t}$$

$$= -e^{4t} + e^{4t}t + e^{3t}$$

240) a)

$$\ln \frac{s^2+1}{(s-1)^2}$$

verwende $-\frac{d}{ds} \cdot F(s) = \mathcal{L}\{t \cdot f(t)\}$

$$-\frac{d}{ds} \ln \frac{s^2+1}{(s-1)^2} = \frac{\cancel{(s-1)^2}}{(s^2+1)} \cdot \frac{2s(s-1)^2 - 2\cancel{(s-1)}(s^2+1)}{((s-1)^2)^2}$$

$$= \frac{2s(s-1) - 2\cdot(s^2+1)}{(s^2+1)(s-1)} = \frac{2s^2 - 2s - 2s^2 - 2}{(s^2+1)(s-1)}$$

$$= -\frac{2(s+1)}{(s^2+1)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$A(s^2+1) + (Bs+C)(s-1) = -2s-2$$

$$s^2\underbrace{(A+B)}_{0} + x\underbrace{(C-B)}_{-2} + \underbrace{A-C}_{-2} = -2s-2$$

$A+B=0$

$A=-B$

$-B-C=-2$

$C=2-B$

$2-B-B=-2$

$B=2 \Rightarrow C=0 \Rightarrow A=2$

$$\frac{2}{s-1} - \frac{2s}{s^2+1} \Rightarrow 2\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= 2\cdot e^t - 2\cdot\cos(t) = f(t)\cdot t \Rightarrow \frac{2(e^t-\cos(t))}{t} = x(t)$$

240)

b) $\frac{e^{-2s} - e^{-4s}}{s} = e^{-2s} \cdot \frac{1}{s} - e^{-4s} \cdot \frac{1}{s}$

$\mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{1}{s} - e^{-4s} \cdot \frac{1}{s} \right\}$

$u(t-2) - u(t-4)$

$u(t) \Rightarrow$ Heaviside funktion