GMA

Grundlagen
Mathematik und Analysis

Reelle Funktionen

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Data Analytics and Computing

GMA - Inhalt

Reelle Funktionen $f: \mathbb{R} \rightarrow \mathbb{R}$

- Relationen
- Funktionen
- Eigenschaften von Funktionen
- Stetigkeit
- Differenzierbarkeit
- Integrierbarkeit

Relationen

Kartesisches Produkt und Graph

Kartesisches Produkt

$$A \times B \doteqdot \{(a, b) \mid a \in A, b \in B\}$$

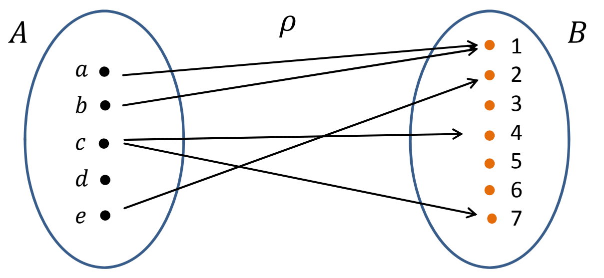
Graph

$$G \subseteq A \times B$$

Relation

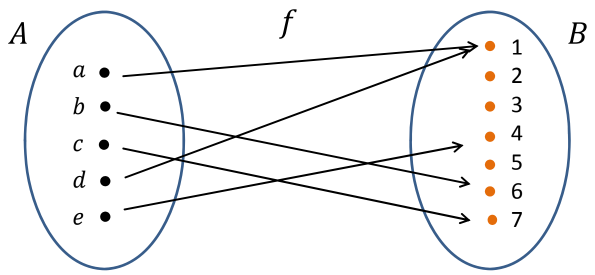
Jeder Graph bestimmt eine Relation

Relationen



$$\rho: A \rightarrow B, R = \{(a, 1), (b, 1), (c, 4), (c, 7), (e, 2)\} \subset A \times B$$

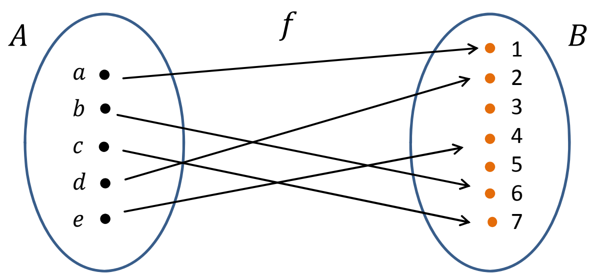
Funktionen



Funktion: $\forall a \in A: \ \exists! \ Z \in B: \ f(a) = Z$

Funktionen

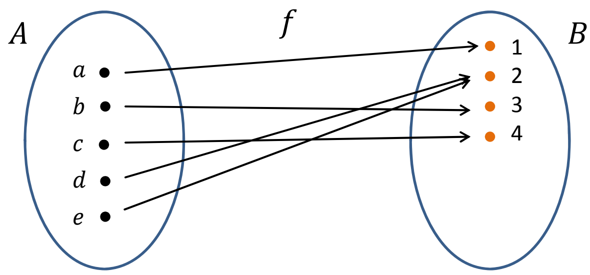
Injektive Funktion



$$f(x) = f(y) \quad \Leftrightarrow \quad x = y$$

Funktionen

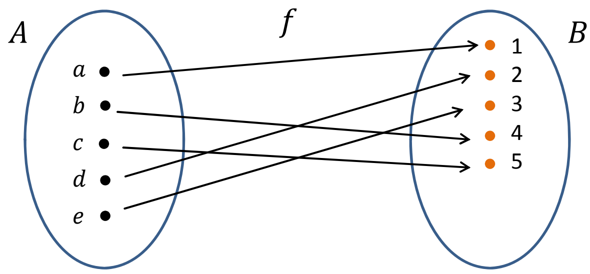
Surjektive Funktion



$$\forall\, Z \in B:\ \exists\, x \in A:\ \ f(x) = Z$$

Funktionen

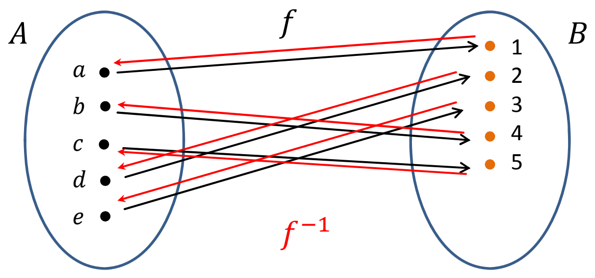
Bijektive Funktion



bijektiv = injektiv und surjektiv gleichzeitig

Funktionen

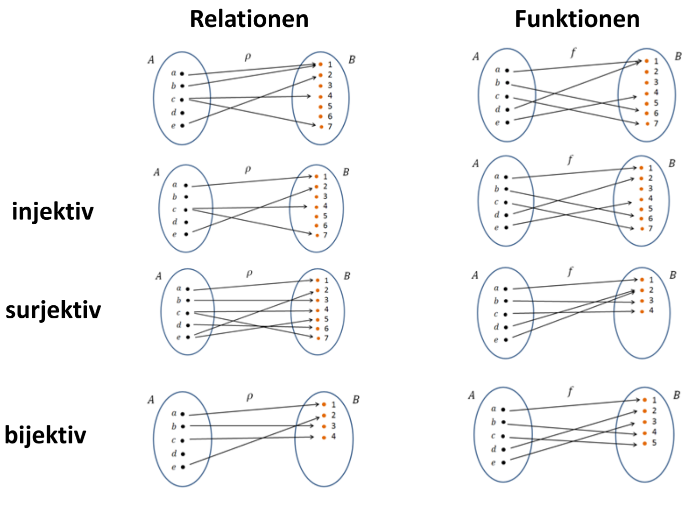
Bijektive Funktion



f bijektiv $\Leftrightarrow$ f invertierbar, $\exists f^{-1}$

Funktionen und Relationen

Vergleich



Funktionen

Identitätsfunktion $id_A : A \to A,\ a \mapsto a,\ \forall a \in A$

Projektion $P_x : \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (x,y) \mapsto x,\ \forall (x,y) \in \mathbb{R}^2$

$P_x : \mathbb{R}^n \to \mathbb{R}^k,\ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$

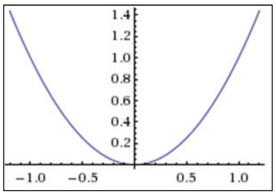
Betragsfunktion $f : \mathbb{R} \to \mathbb{R}_0^+,\ x \mapsto |x| = \begin{cases} x & \text{falls } x \geq 0 \\ -x & \text{falls } x < 0 \end{cases}$

Addition $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (a,b) \in \mathbb{R} \times \mathbb{R} \mapsto a + b \in \mathbb{R}$

$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ (a,b) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto a + b \in \mathbb{R}^n$

Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

Graph



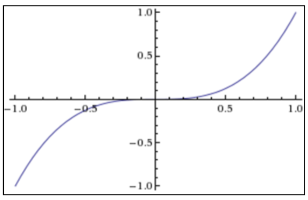
$f(x) = x^2$

Graph der Funktion f

$f : A \to B,\ a \mapsto b = f(a)$

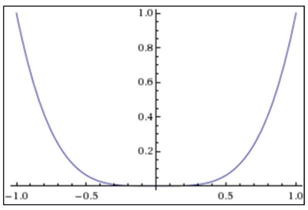
$G = \{(a, f(a)) \mid a \in A,\ f(a) \in B\} \subseteq A \times B$

$G = \{(x, x^2) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$



$f(x) = x^3$

$f(x) = x^4$



http://www.wolframalpha.com/

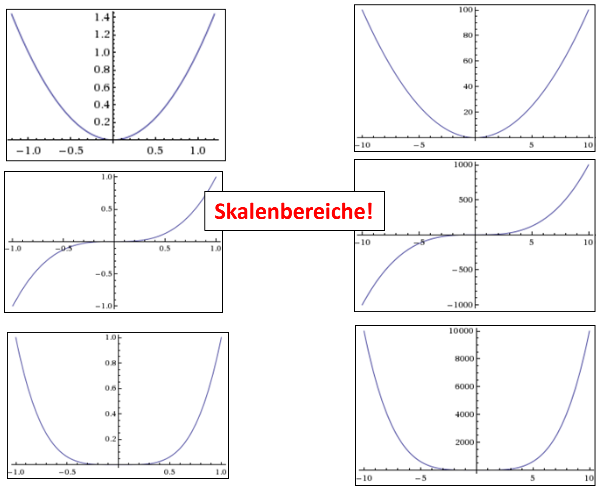
Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

Graph

x^2

x^3

x^4



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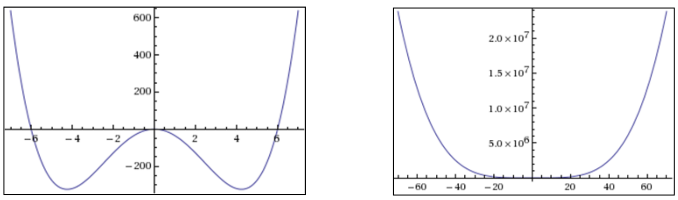
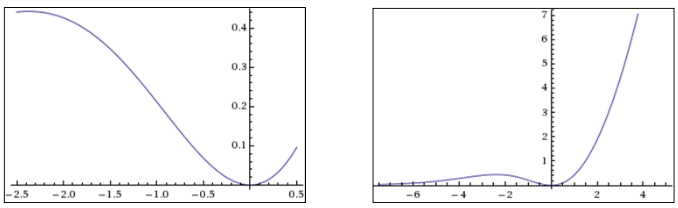
Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

Graph



$x^4 - 36\,x^2$ $\qquad$ $x^4 - 36\,x^2$



$\dfrac{x^2}{2 + e^{-x}}$ $\qquad$ $\dfrac{x^2}{2 + e^{-x}}$

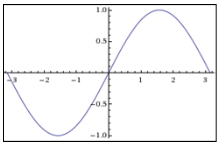
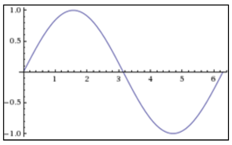
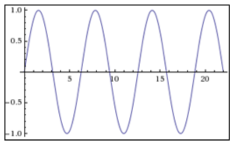
http://www.wolframalpha.com/

Reelle Funktionen $f\colon \mathbb{R} \to \mathbb{R}$

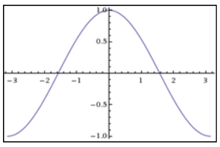
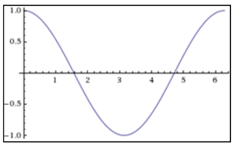
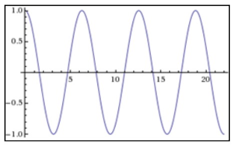
Periodische Funktionen

Sinus

Periodische Funktion: $f(x) = f(x + p)$, Periodenlänge p

$-\pi$ π 0 π 0 7π

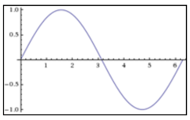




Cosinus

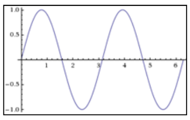
http://www.wolframalpha.com/

Reelle Funktionen $f\colon \mathbb{R} \to \mathbb{R}$

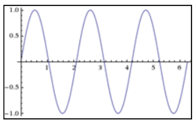
Frequenz



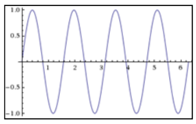
$\sin(x)$



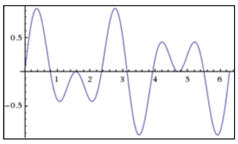
$\sin(2x)$



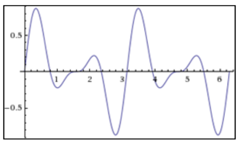
$\sin(3x)$



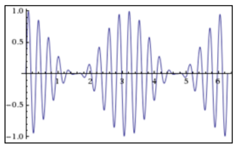
$\sin(4x)$



$\sin(4x)\cos(x)$



$\sin(4x)\cos^2(x)$



$\sin(20x)\cos^2(x)$

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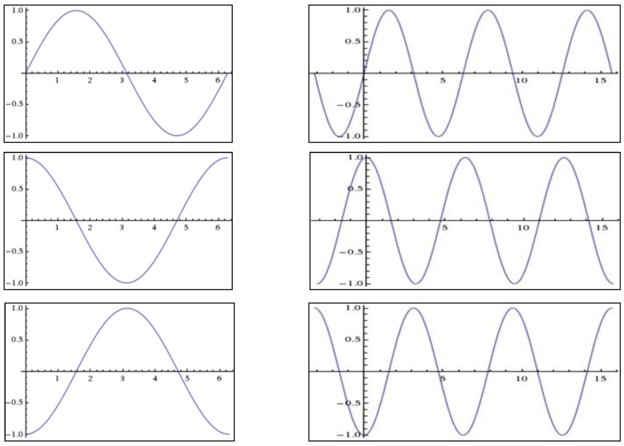
Reelle Funktionen $f\colon \mathbb{R} \to \mathbb{R}$

Phasenverschiebung

$\sin(x)$

$\sin(x + \frac{\pi}{2})$

$\sin(x - \frac{\pi}{2})$

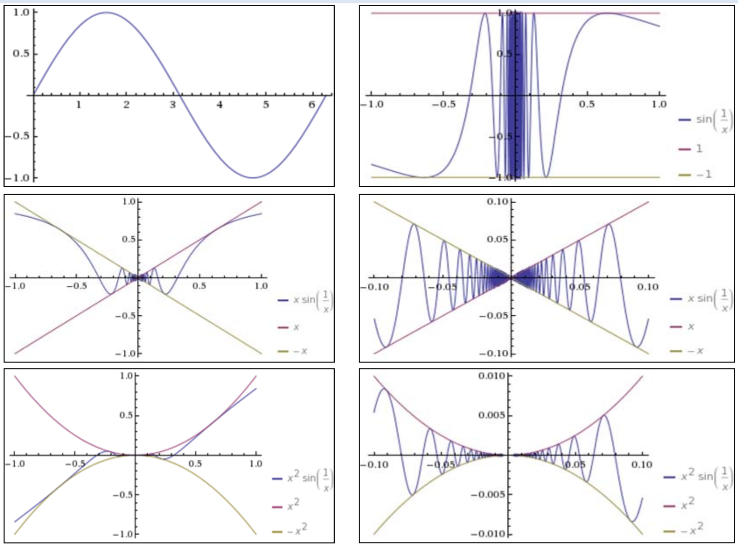


$[0,2\pi]$ $[-\pi, 5\pi]$

http://www.wolframalpha.com/

Reelle Funktionen $f\colon \mathbb{R} \to \mathbb{R}$

Dominanz



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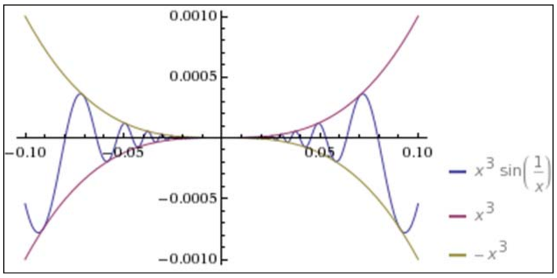
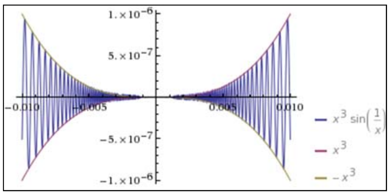
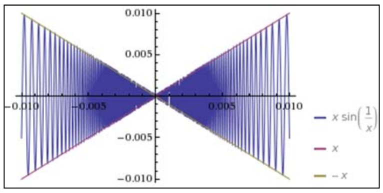
Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$







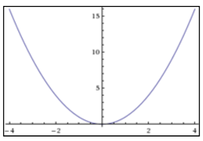
plot[x^3*{sin(1/x),1,-1},{x,-.01,.01}]

Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

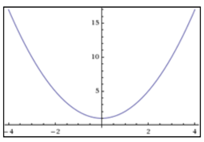
x^2

$f(x)$



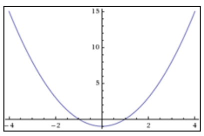
$x^2 + 1$

$f(x) + 1$



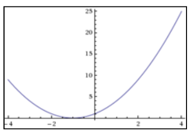
$x^2 - 1$

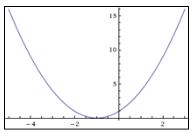
$f(x) - 1$



$(x+1)^2$

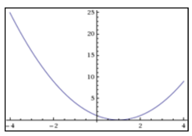
$f(x+1)$

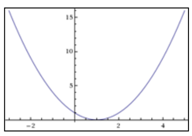




$(x-1)^2$

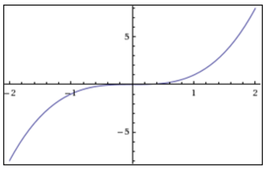
$f(x-1)$



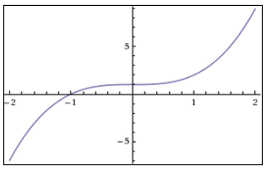


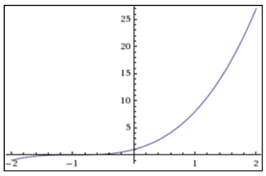
Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

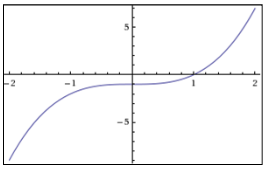
$f(x) = x^3$

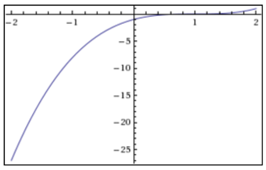


„Verschiebungen“



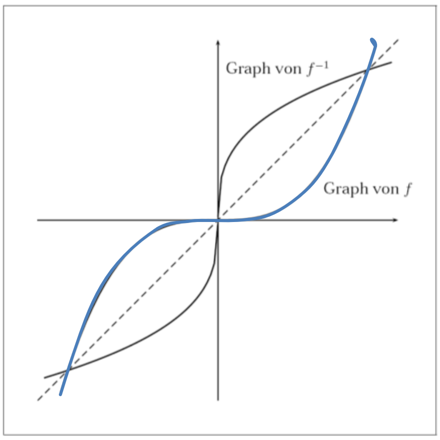






Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

Umkehrfunktion, Inverse



umkehrbar $\Leftrightarrow$ bijektiv

bijektiv=injektiv+surjektiv

injektiv

$f: A \to B,\ a \mapsto b = f(a)$

$f(a) = f(b) \Leftrightarrow a = b$

surjektiv

$f: A \to B,\ a \mapsto b = f(a)$

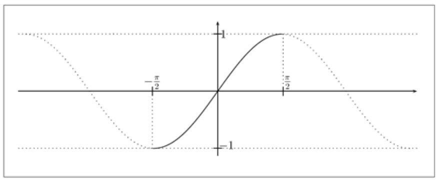
$\forall b \in B \quad \exists a \in A: b = f(a)$

Umkehrfunktion

$f^{-1}: B \to A,\ b = f(a) \mapsto a$

Reelle Funktionen $f\colon \mathbb{R} \to \mathbb{R}$

Umkehrfunktion, Inverse

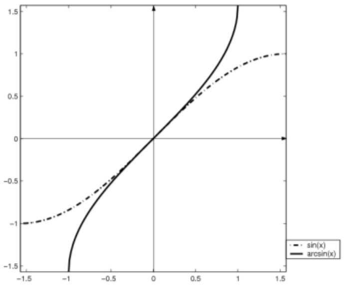


$f : A \to B,\ a \mapsto b = f(a)$

$H \subseteq A$ eine Teilmenge von A

Einschränkung von f auf H

$f_{|H} : H \to B,\ a \mapsto b = f(a)$



Sinus ist auf $[-\frac{pi}{2}, \frac{\pi}{2}]$ bijektiv

Umkehrfunktion arcsin

sin: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$

arcsin: $[-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Reelle Funktionen $f\colon \mathbb{R} \to \mathbb{R}$

Monotonie

Eine Funktion $f : A \to B,\ a \mapsto b = f(a)$ heißt *monoton wachsend*, falls gilt

$$f(x) \leq f(y) \qquad \forall x < y\,.$$

Sie heißt *streng monoton wachsend*, wenn gilt

$$f(x) < f(y) \qquad \forall x < y\,.$$

Eine Funktion $f : A \to B,\ a \mapsto b = f(a)$ heißt *monoton fallend*, falls gilt

$$f(x) \geq f(y) \qquad \forall x < y\,.$$

Sie heißt *streng monoton fallend*, wenn gilt

$$f(x) > f(y) \qquad \forall x < y\,.$$

Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

Verknüpfung, Komposition

Funktionen

$$f : A \to B \text{ und } g : B \to C$$

zusammengesetzte Funktion

$$h = g \circ f : A \to C, \; a \in A \mapsto c = g(f(a)) \in C$$

kommutatives Diagramm

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Spezielle reelle Funktionen

Geraden:	$x \mapsto kx + d$ k heißt Steigung oder Anstieg	$\mathbb{R} \to \mathbb{R}$
Potenzfunktionen:	$x \mapsto x^n \quad n \in \mathbb{R}$	$D \to W$
Wurzelfunktionen:	$x \mapsto \sqrt[n]{x} = x^{1/n} \quad n \in \mathbb{N}$ $D = W = \mathbb{R}_0^+$ falls n gerade $D = W = \mathbb{R}$ falls n ungerade	$D \to W$
Polynome:	$x \mapsto p(x) = \sum_{k=0}^{n} a_k x^k$	$\mathbb{R} \to \mathbb{R}$
Rationale Funktionen:	$x \mapsto \frac{p(x)}{q(x)}$ $p(x), q(x)$ Polynome $D = \mathbb{R} \setminus \{\text{Nullstellen von } q\}$	$D \to \mathbb{R}$

Spezielle reelle Funktionen

Exponentialfunktion: $x \mapsto \exp(x) = e^x \qquad \mathbb{R} \to \mathbb{R}^+$

$x \mapsto a^x \quad a \in \mathbb{R}^+ \qquad \mathbb{R} \to \mathbb{R}^+$

Logarithmusfunktion: $x \mapsto \log(x) = \ln(x) \qquad \mathbb{R}^+ \to \mathbb{R}$

Inverse Funktion zur *Exponentialfunktion*

$x \mapsto \log_a(x) \quad a \in \mathbb{R}^+ \qquad \mathbb{R}^+ \to \mathbb{R}$

Winkelfunktionen: $x \mapsto \sin(x) \qquad \mathbb{R} \to [-1,1]$

$x \mapsto \cos(x) \qquad \mathbb{R} \to [-1,1]$

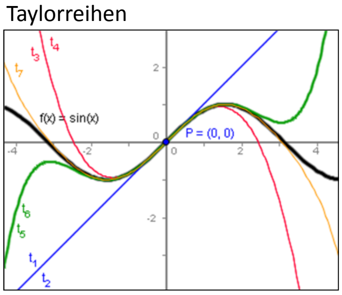
$\tan(x),\ \cot(x)$

$\arcsin(x),\ \arccos(x)$

Reelle Funktionen $f: \mathbb{R} \to \mathbb{R}$

Reihenentwicklungen

Taylorreihen



http://teacher.eduhi.at/alindner/Dyn_Geometrie/Taylor/

Fourierreihen