Blackboard Exercises (Summer Term 2012)

This document contains the exercises from the lecture's first logic part which were solved at the blackboard.

1 Recap: How to prove "A iff B"?

Usually, the proof is split into two parts, namely to show "$A \implies B$" and "$B \implies A$" (sometimes written as "$A \impliedby B$"). There are at least two principal possibilities to show "$A \implies B$". In the first one, we assume A and derive B (possibly using other lemmata, theorems, etc.). In the second possibility, we assume $\neg B$ and derive $\neg A$. The correctness of the latter approach follows from the correctness of the former together with the fact that $(A \to B)$ and $(\neg B \to \neg A)$ are logically equivalent.

2 Show that $\forall$ distribution is correct (student's question)

We show that $\big((\forall x\, \phi(x)) \wedge (\forall x\, \psi(x))\big) \equiv \big(\forall x\, (\phi(x) \wedge \psi(x))\big)$ holds. We show that each model of the right formula is also a model of the left formula and vice versa.

$\implies$: Let U be an arbitrary domain and I_α an arbitrary model of $(\forall x\, \phi(x)) \wedge (\forall x\, \psi(x))$. Then

$$\begin{aligned} & I_\alpha \models (\forall x\, \phi(x)) \wedge (\forall x\, \psi(x)) \\ \text{iff} \quad & I_\alpha \models \forall x\, \phi(x) \text{ and } I_\alpha \models \forall x\, \psi(x) \\ \text{iff} \quad & I_\alpha \models \phi(c) \text{ for all } c \in U \text{ and } I_\alpha \models \psi(d) \text{ for all } d \in U \end{aligned}$$

Hence, ϕ and ψ hold exactly for the same elements under I_α and therefore $I_\alpha \models \phi(c) \wedge \psi(c)$ holds for all $c \in U$. Finally, $I_\alpha \models \forall x\, (\phi(x) \wedge \psi(x))$ holds by the semantics of $\forall$.

$\impliedby$: Similar to the first direction.

3 Proof of the Deduction Theorem

In the lecture, we discussed the following theorem:

$$\varphi \models \psi \qquad \text{if and only if} \qquad \models \varphi \to \psi.$$

The proof for the propositional variant is as follows.

$\implies$: Assume $\not\models \varphi \to \psi$. Then there exists an interpretation I with $I \not\models \varphi \to \psi$, i.e., $I \models \varphi$ and $I \not\models \psi$ by the semantics of $\to$. But then $I \in \mathit{Mod}(\varphi)$, $I \notin \mathit{Mod}(\psi)$ and $\mathit{Mod}(\varphi) \not\subseteq \mathit{Mod}(\psi)$. Therefore $\varphi \not\models \psi$ holds.

$\impliedby$: Assume $\varphi \not\models \psi$. Then, $\mathit{Mod}(\varphi) \not\subseteq \mathit{Mod}(\psi)$ and there exists an interpretation I with $I \in \mathit{Mod}(\varphi)$ and $I \notin \mathit{Mod}(\psi)$. Therefore $I \models \varphi$, but $I \not\models \psi$. Hence $I \not\models \varphi \to \psi$.

What has to be changed in the above proof for the first-order variant of the deduction theorem?