Problem Set 2

Problem 2.1 Show that if x is Cauchy distributed with parameter α, then $\mathsf{y} = 1/\mathsf{x}$ is Cauchy distributed with parameter $1/\alpha$.

Problem 2.2 Define a random variable by $\mathsf{y} = \sin(\mathsf{x})$, where x is uniform distributed between $-\pi$ and $+\pi$. Calculate and sketch the pdf of y.

Problem 2.3 Let $\mathsf{x} \sim \mathcal{N}(\mu_\mathsf{x}, \sigma_\mathsf{x}^2)$ be a Gaussian random variable with mean $\mu_\mathsf{x} = 4$ and variance $\sigma_\mathsf{x}^2 = 5$.

a) Calculate the probability that x is in the interval $[-2, 3]$.

b) A random variable y is obtained from x via the clipping operation $g(\cdot)$ as follows:

$$\mathsf{y} = g(\mathsf{x}) = \begin{cases} a, & \mathsf{x} \le 2 \\ \mathsf{x}, & 2 < \mathsf{x} < 6 \\ b, & \mathsf{x} \ge 6 \end{cases}.$$

Find expressions for a and b such that the mean square error $\mathrm{E}\{(\mathsf{x} - \mathsf{y})^2\}$ is minimized and evaluate these expressions numerically. Find and sketch the pdf of y.

c) The random variable y is now quantized (binned) with Q levels (bins) yielding a new random variable z, i.e.,

$$\mathsf{z} = q(\mathsf{y}) = k \qquad \text{if} \quad \mathsf{y} \in [g_{k-1}, g_k] \subset \mathbb{R},\ k = 1, 2, \ldots, Q,$$

where g_i denotes the ith quantization boundary and $g_0 = -\infty$, $g_Q = \infty$. For $Q = 3$, find the remaining g_i such that $p_\mathsf{z}(z) = 1/Q$.

Problem 2.4 The characteristic function for a zero-mean Gaussian random variable x is

$$\Phi_\mathsf{x}(\omega) = e^{-\sigma_\mathsf{x}^2 \omega^2/2}.$$

Calculate all moments $m_\mathsf{x}^{(n)}$ of x using $\Phi_\mathsf{x}(\omega)$ as function of n and σ_x.

Hint: $e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}$