This is the seventh homework assignment. Students should tick in TUWEL problems they have solved and upload their detailed solutions by **20:00** on Monday **December 5th, 2022**.

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(1) **z-test (without R)**
In the context of a two-sided $z$-test let $\bar{x} = \frac{4}{3}$, $\sigma = 1$, $n = 9$ and $H_0 : \mu = \frac{2}{3}$.
The $\alpha$-quantiles $q_\alpha$ of $\mathcal{N}(0, 1)$ are as follows

| $\alpha$ | 0.005 | 0.010 | 0.015 | 0.020 | 0.025 | 0.030 | 0.035 | 0.040 |
|---|---|---|---|---|---|---|---|---|
| $q_\alpha$ | −2.576 | −2.326 | −2.170 | −2.054 | −1.960 | −1.881 | −1.812 | −1.751 |

How do you decide for a significance level of 1%, (b) 3%, (c) 6% and (d) 10% by only using the information above? Justify your answers.

(2) **Interpretation of test results**
In the context of a statistical test at significance level $\alpha$, the test statistic lies in the rejection region. Comment on the following statements.

(a) The null hypothesis is rejected at the $\alpha$-level
(b) The null hypothesis is rejected at the $\alpha/2$-level.
(c) The null hypothesis is rejected at the $2\alpha$-level.
(d) The $p$-value was at least $\alpha$.
(e) The null hypothesis was significant.
(f) The null hypothesis is not true.
(g) The null hypothesis is probably not true.

(3) **Test power in the $z$-test**
Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_1 \sim \mathcal{N}(\mu, \sigma^2)$, and $H_0 : \mu = \mu_0$.

(a) Compute the test power of the left-sided $z$-test. Express it through the cdf of the standard normal distribution, depending on $\mu_0, \mu, \sigma, n$ and the significance level $\alpha$.
(b) Comment on the impact of $\mu_0, \mu, \sigma, n$ and $\alpha$ on the test power.

(4) **Type I error and Type II error**

Let $X_1, \ldots, X_{16}$ be i.i.d. random variables with $X_1 \sim \mathcal{N}(0, 4)$. Assume that for a realization it holds $\bar{x} = 4$. In the context of a right-sided $z$-test, let $H_0 : \mu = 2$ and the rejection area $R = [3, +\infty)$. Which of the following statements are correct?

(a) We will commit a Type I error

(b) We will commit a Type II error

(c) We will not commit a Type II error.

(d) If we increase the significance level of the test, then we obtain a higher test power

*Hint:* The expectation of $X_1$ is fixed at zero. Is the null hypothesis true?

*Remark:* Note that this is a theoretical consideration of the $z$-test. Here, the distribution of $X_1$ is fixed (as the expectation is zero), while usually we assume the expectation (resp. *the population*) to be unknown. In this theoretical consideration we know whether the null hypothesis is true and we are thus able to make statements about the errors. From a practical point of view, the setup is not useful, because the reason to perform a statistical test is the fact that the population of interest is unknown.

(5) **Law of large numbers**

Visualize the law of the large numbers assuming exponential distributed random variables. Let $X_1, X_2, \ldots$ be i.i.d. random variables, with $X_1 \sim \exp(\lambda)$.
For $m = 1, 2, \ldots$ the mean of the first $m$ random variables is given as

$$\bar{X}_m = \frac{1}{m} \sum_{i=1}^{m} X_i, \qquad m \in \mathbb{N}.$$

Further, for $n \in \mathbb{N}$ a sequence of means is given as $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$. Assume $\lambda = 0.5$ and $n = 200$ and plot 20 realized sequences of means in one graph. Mark the expectation of $X_1$. Comment on the obtained result.