Statistics and Probability: HW 1 - Problem 1

Friday

Duedate: 13.10.2020

Car licence plates

A certain state's car licence plates have three letters of the alphabet followed by a three-digit number.

Problem 1a)

How many different licence plates are possible if all three-letter sequences are permitted and any number from 000 to 999 is allowed?

Solution

First, there are 26 letters in the alphabet (asuming non-speical characters are not allowed and all letters are uppercase). The same is obvious for the digits, 10 posibilities for each position.

$$26^3 * 10^3 = 17576 * 1000 = 17\,576\,000 \quad \square \tag{1}$$

Problem 1b)

Mary withnessed a hit-and-run accident. She knows that the first letter on the licence plate of the offender's car was a M, that the second letter was an A or H, and that the last number was a 7. How many state's licence plates fit this description?

Solution

We start be creating a table to show the posibilities we have for each position in the licence plate.

Position	1	2	3	4	5	6
What we know	M	A or H	*any letter*	*any digit*	*any digit*	7
Posibilities	1	2	26	10	10	1

Now we can multiply the posibilities for each position to get the total number of posible licence plates.

$$1 * 2 * 26 * 10 * 10 * 1 = 5\,200 \quad \square \tag{2}$$

Statistics and Probability: HW 1 - Problem 2

Friday

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Symphony orchestra program

A symphony orchestra has in its repertoire 30 Haydn symphonies, 15 modern works, and 9 Beethoven symphonies. Its program always consists of a Haydn symphony followed by a modern work, and then a Beethoven symphony.

Problem 2a)

How many different programs can it play?

Solution

$$30 * 15 * 9 = 4\,050 \quad \square \tag{1}$$

Problem 2b)

How many different programs are there if the three pieces can be played in any order?

Solution

First we can look at the possibilities to arange the three pieces. This permutation can be calculated with 3!. We know from 2a that there are 4050 possibe programs for each of these arangements so the solution is:

$$3! * 4\,050 = 24\,300 \quad \square \tag{2}$$

Problem 2c)

Assume that each piece cannot be played more than once. How many different three- piece programs are there if more than one piece from the same category can be played and they can be played in any order?

Solution

Here we want to select k elements from n elements, while each element can only selected be once and the order in which we select them in relevant. The number of possibilities can be calucated with: $\frac{n!}{(n-k)!}$

So, if we fill in the numbers from the problem we get the following result:

$$
\begin{aligned}
n &= 30 + 15 + 9 = 54 \\
k &= 3 \\
\frac{n!}{(n-k)!} &= \frac{54!}{(54-3)!} = 148\,824 \quad \square
\end{aligned}
\tag{3}
$$

Statistics and Probability: HW 1 - Problem 3

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Poker game

A deck of 52 cards has 13 ranks (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A) and 4 suits (♣♦♥♠). A poker hand is a set of 5 cards randomly chosen from a deck of 52 cards.

Problem 3a)

A full house in poker is a hand where three cards share one rank and two cards share another rank. How many ways are there to get a full-house? What is the probability of getting a full-house?

Solution

First we need to calculate all possible hands there are. Simplified we want to count the number of subsets, while ignoring the order. The number of all combinations can be calculated with:

$$\binom{n}{k} = \binom{52}{5} = 2\,598\,960 \tag{1}$$

Next, we calculate how many hands result in a full-house. We can think of it as choosing a rank from which we draw 3 cards and then choosing a different rank from which we draw 2 cards. (There are always 4 cards in each rank)

$$\binom{13}{1} * \binom{4}{3} * \binom{12}{1} * \binom{4}{2} = 3\,744 \tag{2}$$

Now, all we have to do is to devide the desired outcomes by number of possible outcomes.

$$\frac{3\,744}{2\,598\,960} \approx 0,00144 = 1,44 * 10^{-3} \quad \square \tag{3}$$

Problem 3b)

A royal flush in poker is a hand with ten, jack, queen, king, ace in a single suit. What is the probability of getting a royal flush?

Solution

Since a royal flush fills a full hand **and** all cards must be of the same suit **and** there are only 4 suits, it is obvious that there are only 4 hands that result in a royal flush.

$$\frac{4}{2\,598\,960} \approx 0.00000154 = 1.54 * 10^{-6} \quad \square \tag{4}$$

Statistics and Probability: HW 1 - Problem 4

Friday

Duedate: 13.10.2020

Student athletes

A random sample of 400 college students was asked if college athletes should be payed. The following table gives a two-way classification of the responses.

	Should be paid	Should not be paid
Student athlete	90	10
Student nonathlete	210	90

Problem 4a)

If one student is randomly selected from these 400 students, find the probability that this student

i. is in favor of paying college athletes

ii. is an athlete and favors paying student athletes

iii. is a nonathlete or is against paying students athletes

Solution for i

There are 400 students in total, of which 300 are in favor of paying athletes. Now we can divide the desireable outcomes by the number possible ones and get:

$$\frac{300}{400} = 0,75 \quad \square \tag{1}$$

Solution for ii

Only 90 students are athletes and in favor of paying.

$$\frac{90}{400} = 0,225 \quad \square \tag{2}$$

Solution for iii

To solve this problem we need to apply the Inclusion-Exclusion Principle. Let A be the set of students that are nonathletes and B is the set of students that are against paying students.

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 300 + 100 - 90 \\ &= 310 \end{aligned} \tag{3}$$

Now all we have to do is to divide them by the total number of students.

$$\frac{310}{400} = 0,775 \tag{4}$$

Problem 4b)

Are the events student athlete and should be paid mutually exclusive? Justify your answer.

Solution

We describe the set of student athletes as A and the set of should be paid as B. A and B would be mutually exclusive if $A \cap B = \emptyset$ would be true. However, from the table we know that $|A \cap B| = 90$ which is contrary to the definition of mutually exclusive. Therefore the two sets are not mutually exclusive.

Statistics and Probability: HW 1 - Problem 5

Friday

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Coin game

Two players, A and B, alternately and independently flip a coin and the first player to obtain a head wins. Assume player A flips first.

Problem 5a)

If the coin is fair, what is the probability that A wins?

Solution

If a describes the posibility that A winns and b the winn of B, than we can create the following equation.

$$a = \frac{1}{2} + \frac{1}{2} * (1 - a) \tag{1}$$

First we have the probability that A winns imideatly which is $\frac{1}{2}$, or A loses at first but wins in one of the later rounds. This second part can be described with the probability that first A has to lose but in the end B has to lose the game. $(1 - a)$ describes the probability that B will lose the game.

So all we have to do now is to solve this equation:

$$\begin{aligned} a &= \frac{1}{2} + \frac{1}{2} * (1 - a) \\ a &= \frac{1}{2} + \frac{1}{2} - \frac{a}{2} \\ \frac{3a}{2} &= 1 \\ a &= \frac{2}{3} \quad \square \end{aligned} \tag{2}$$

Problem 5b)

Suppose that $P(head) = p$, not necessarily $\frac{1}{2}$. What is the probability that A wins?

Solution

Let a be the probability that A wins. A detailed description as how the equation gets created can be found at the problem at 5a).

$$\begin{aligned} a &= p + (1-p) * (1-a) \\ a &= p + 1 - a - p + ap \\ a &= 1 - a + ap \\ 2a - ap &= 1 \\ a * (2-p) &= 1 \\ a &= \frac{1}{2-p} \quad \square \end{aligned} \tag{3}$$

Statistics and Probability: HW 1 - Problem 6

Friday

Duedate: 13.10.2020

Binominal coefficients

Problem 6a)

Prove that $\binom{n}{j} = \binom{n}{n-j}$ holds for $n \in \mathbb{N}$ and $0 \leq j \leq n$.

Solution

$$
\begin{aligned}
\binom{n}{j} &= \frac{n!}{j! * (n-j)!} \\
&= \frac{n!}{(n-j)! * (n-n+j)!} \\
&= \frac{n!}{(n-j)! * (n-(n-j))!} \\
&= \binom{n}{n-j} \quad \square
\end{aligned} \tag{1}
$$

Problem 6b)

Prove that $\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$ holds for $n \in \mathbb{N}$ and $0 < j < n$.

Solution

$$\begin{aligned}
\binom{n-1}{j} + \binom{n-1}{j-1} &= \frac{(n-1)!}{j! * (n-1-j)!} + \frac{(n-1)!}{(j-1)! * (n-1-(j-1))!} \\
&= \frac{(n-1)!}{j! * (n-1-j)!} + \frac{(n-1)!}{(j-1)! * (n-j)!} \\
&= \frac{(n-1)! * (n-j)}{j! * (n-j)!} + \frac{j * (n-1)!}{j! * (n-j)!} \\
&= \frac{(n-1)! * (n-j+j)}{j! * (n-j)!} \\
&= \frac{(n-1)! * n}{j! * (n-j)!} \\
&= \frac{n!}{j! * (n-j)!} \\
&= \binom{n}{j} \quad \square
\end{aligned} \tag{2}$$

Problem 6c)

Find integers n and r such that the equation $\binom{13}{5} + 2\binom{13}{6} + \binom{13}{7} = \binom{n}{r}$ is true.

Solution

$$\begin{aligned}
\binom{n}{r} &= \binom{13}{5} + \binom{13}{6} + \binom{13}{6} + \binom{13}{7} \\
\binom{n}{r} &= \binom{14}{6} + \binom{14}{7} \\
\binom{n}{r} &= \binom{15}{7} \quad \square
\end{aligned} \tag{3}$$