Kronecker Algebra

Computer Systems

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1. Kronecker Product
2. Verifying Programs
3. The Kronecker Sum

Kronecker Product

Kronecker Product of Finite State Machines (FSMs)

Definition

Given an m-by-n matrix A and a p-by-q matrix B, their Kronecker product denoted by $A \otimes B$ is a mp-by-nq block matrix defined by

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{pmatrix}.$$

Kronecker Product

Example

For example, if

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix},$$

then

$$A \otimes B = \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,1}b_{1,3} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} & a_{1,2}b_{1,3} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,1}b_{2,3} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} & a_{1,2}b_{2,3} \\ a_{1,1}b_{3,1} & a_{1,1}b_{3,2} & a_{1,1}b_{3,3} & a_{1,2}b_{3,1} & a_{1,2}b_{3,2} & a_{1,2}b_{3,3} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,1}b_{1,3} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} & a_{2,2}b_{1,3} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,1}b_{2,3} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} & a_{2,2}b_{2,3} \\ a_{2,1}b_{3,1} & a_{2,1}b_{3,2} & a_{2,1}b_{3,3} & a_{2,2}b_{3,1} & a_{2,2}b_{3,2} & a_{2,2}b_{3,3} \end{pmatrix}.$$

Kronecker Product

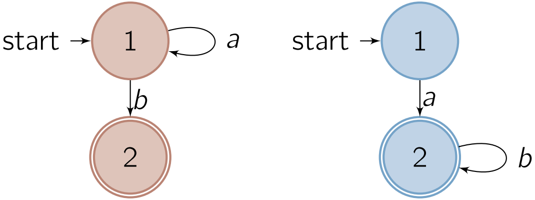


Abbildung: FSMs A (left) and B (right)

Kronecker Product

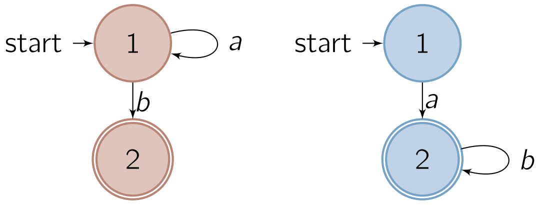


Abbildung: FSMs A (left) and B (right)

Example

matrices

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \; A \otimes B = \begin{pmatrix} \cdot & aa & \cdot & ba \\ \cdot & ab & \cdot & bb \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Kronecker Product

Initial and Final States

Let S_A and S_B be the initial state vectors of the operands, F_A and F_B their final state vectors. Then the initial vector of the Kronecker product is given by $S_A \otimes S_B$ and its final vector is $F_A \otimes F_B$.

Example

$S_A = (1, 0)$, $S_B = (1, 0)$, $F_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $F_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus $S_{A \otimes B} = (1, 0, 0, 0)$ and

$$F_{A \otimes B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which simply state that the initial state of the Kronecker product FSM $A \otimes B$ is state 1 and its final state is state 4.

Kronecker Product for FSMs

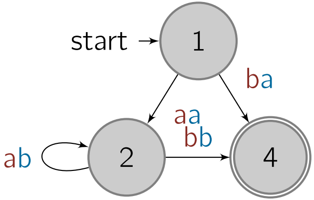
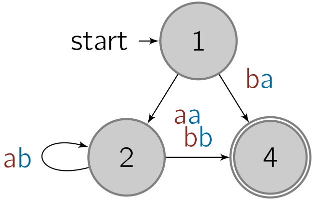


Abbildung: Graphical Representation of $A \otimes B$

Kronecker Product for FSMs



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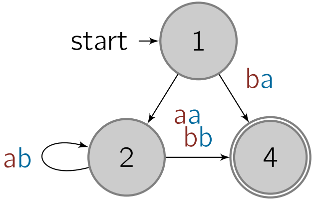
Surprise

matrix has size four

Kronecker Product for FSMs



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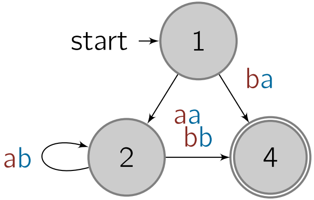


Abbildung: Graphical Representation of $A \otimes B$

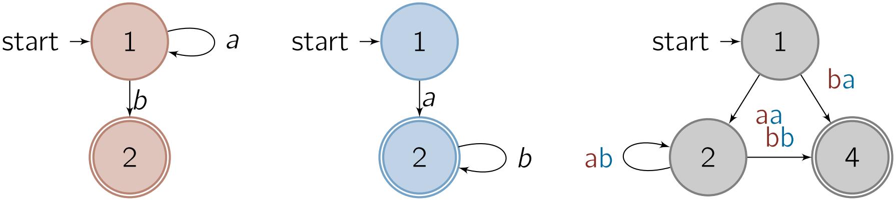
Surprise

matrix has size four

only three states above

State 3 cannot be reached from the initial state 1!

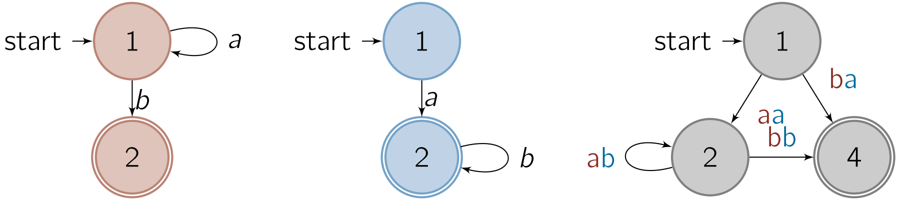
Kronecker Product for FSMs



Both FSMs, *A* and *B* perform their state transitions in *lockstep*.

1. At the beginning, both FSMs are in their initial state. *B* has to proceed to state 2, thereby generating output *a*. *A* has two possible successor states:
   - A. *A* stays in state 1, producing output *a*. This corresponds to state transition $1 \rightarrow 2$ in the product FSM.
   - B. *A* proceeds to its state 2 (output: *b*). This corresponds to state transition $1 \rightarrow 4$ in the product FSM.

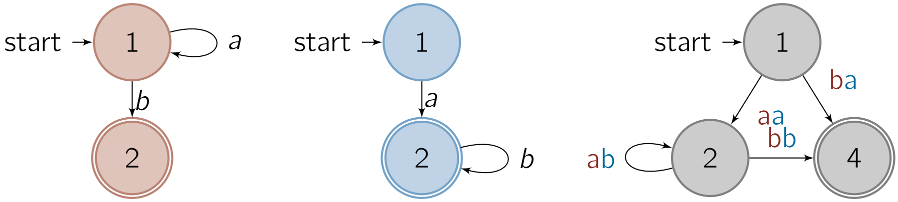
Kronecker Product for FSMs



Both FSMs, *A* and *B* perform their state transitions in *lockstep*.

2 If the previous step was 1A, both FSMs can now produce together an arbitrary number of *ab*-pairs. This corresponds to the transition $2 \rightarrow 2$ in the product FSM. When *A* issues a *b*, *B* also has to produce a *b*. This corresponds to transition $2 \rightarrow 4$ in the product FSM. Then both FSMs are in their final states. *A* cannot do a further state transition and both FSMs and the product FSM terminate.

Kronecker Product for FSMs



Both FSMs, *A* and *B* perform their state transitions in *lockstep*.

3 If the previous step was 1B, both FSMs are in their final states. Since *A* cannot proceed, both FSMs and the product FSM terminate.

Kronecker Product for FSMs

Theorem

Given two FSMs A and B, their Kronecker product FSM generates the same output than A and B do when they execute in lockstep. □

Kronecker Product for FSMs

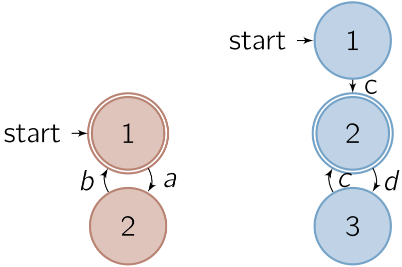


Abbildung: FSMs C (left) and D (right)

Kronecker Product for FSMs

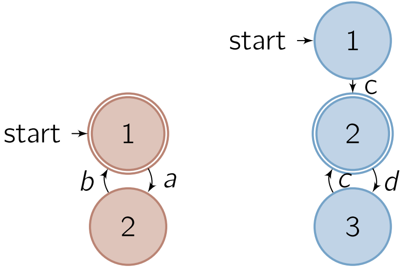


Abbildung: FSMs C (left) and D (right)

C terminates only when it has done an even number of state transitions

Kronecker Product for FSMs

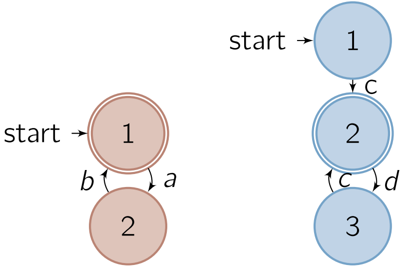


Abbildung: FSMs C (left) and D (right)

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D can terminate only after it has done an odd number of transitions

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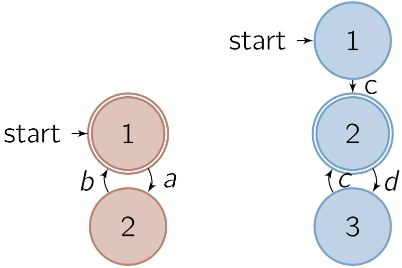

Abbildung: FSMs C (left) and D (right)

C terminates only when it has done an even number of state transitions
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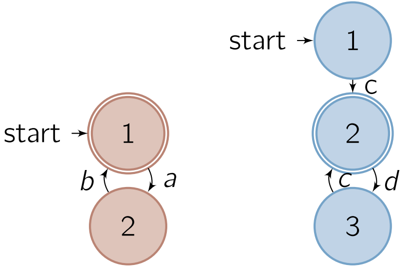
overall output of C ... $(ab)^*$

Kronecker Product for FSMs



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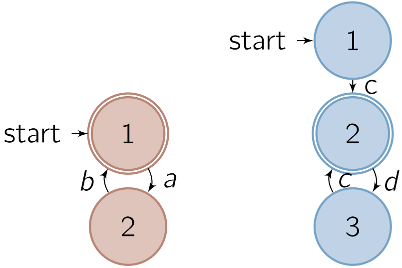
- C terminates only when it has done an even number of state transitions
- D can terminate only after it has done an odd number of transitions
- overall output of C ... $(ab)^*$
- overall output of D ... $c(dc)^*$

Kronecker Product for FSMs



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overall output of D … $c(dc)^*$
the Kronecker product of C and D …

Kronecker Product for FSMs

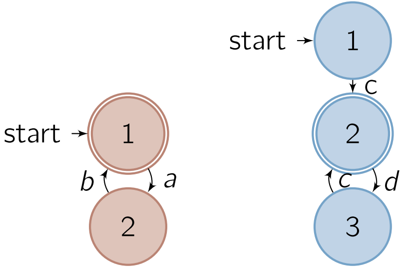


Abbildung: FSMs C (left) and D (right)

C terminates only when it has done an even number of state transitions
D can terminate only after it has done an odd number of transitions
overall output of C $\ldots (ab)^*$
overall output of D $\ldots c(dc)^*$
the Kronecker product of C and D $\ldots 0$.

Kronecker Product for FSMs

$$C = \begin{pmatrix} \cdot & a \\ b & \cdot \end{pmatrix} \text{ and } D = \begin{pmatrix} \cdot & c & \cdot \\ \cdot & \cdot & d \\ \cdot & c & \cdot \end{pmatrix}$$

$$C \otimes D = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & ac & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & ad \\ \cdot & \cdot & \cdot & \cdot & ac & \cdot \\ \cdot & bc & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & bd & \cdot & \cdot & \cdot \\ \cdot & bc & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Kronecker Product for FSMs

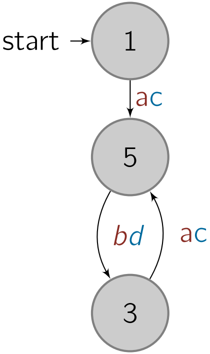
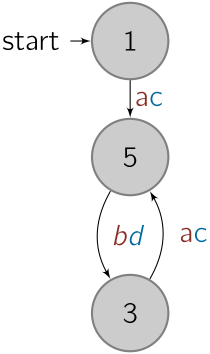


Abbildung: Graphical Representation of $C \otimes D$

the final state 2

Kronecker Product for FSMs



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the final state 2
cannot be reached from the initial state 1.

Verifying Programs

Verifying Programs

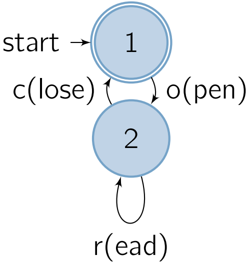

Abbildung: Graphical Representation of File Usage Scenario F

Verifying Programs

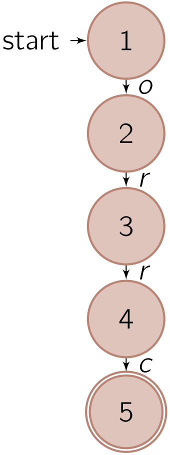

Abbildung: Graphical Representation of File Usage System A

Verifying Programs

$$A \otimes F = \begin{pmatrix} \cdot & o & \cdot & \cdot & \cdot \\ \cdot & \cdot & r & \cdot & \cdot \\ \cdot & \cdot & \cdot & r & \cdot \\ \cdot & \cdot & \cdot & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \otimes \begin{pmatrix} \cdot & o \\ c & r \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & oo & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & oc & or & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & ro & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & rc & rr & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ro & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & rc & rr & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & co \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & cc & cr \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Verifying Programs

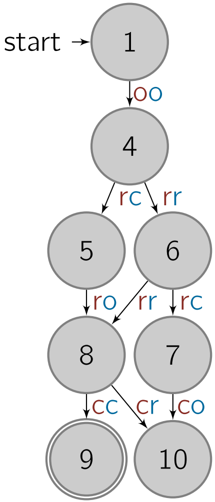

Abbildung: Graphical Representation of $A \otimes F$

Verifying Programs

pairings rc, ro, $\ldots$ do not make sense (lockstep!)

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We assume that $rc = ro = \cdots = 0$
and for simplicity set $oo = o$, $rr = r$, and $cc = c$.

Verifying Programs

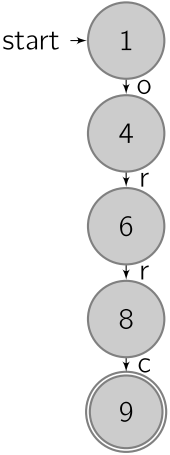

Abbildung: Graphical Representation of “new” $A \otimes F$

Verifying Programs

Figures are very similar

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In fact only the node IDs differ

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In graph theory, an *isomorphism* of graphs G and H is a bijection f between the node sets of G and H such that any two nodes u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. If an isomorphism exists between two graphs, then the graphs are called *isomorphic*. We write $G \simeq H$ in such a case.

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So, clearly the graphs of our example are isomorphic.

Verifying Programs

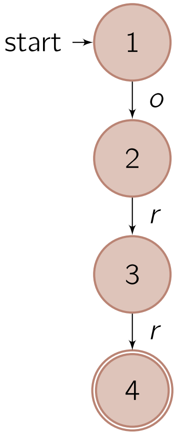

Abbildung: Graphical Representation of File Usage System B

Verifying Programs

$$B = \begin{pmatrix} \cdot & o & \cdot & \cdot \\ \cdot & \cdot & r & \cdot \\ \cdot & \cdot & \cdot & r \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \; B \otimes F = \begin{pmatrix} \cdot & \cdot & \cdot & o & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Verifying Programs

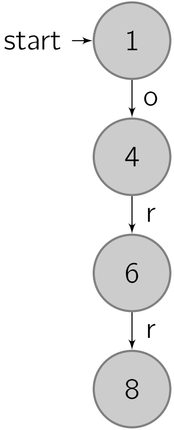

Abbildung: Graphical Representation of $B \otimes F$

Verifying Programs

Isomorphic?

Verifying Programs

Isomorphic?

Definition

An isomorphism of two control flow graphs (CFGs) G and H is a bijection f between the node sets of G and H such that any two nodes u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. In addition, let r be the root node of G. Then $f(r)$ has to be the root node of H. For all final nodes s of G, $f(s)$ have to be final nodes of H, and for all final nodes t of H, $f^{-1}(t)$ have to be final nodes in G. If an isomorphism exists between two CFGs, then the CFGs are called *isomorphic* which we denote by $G \simeq H$.

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With this definition we still have $A \otimes F \simeq A$ but $B \otimes F \not\simeq B$ because there is no final node in this case.

Verifying Programs

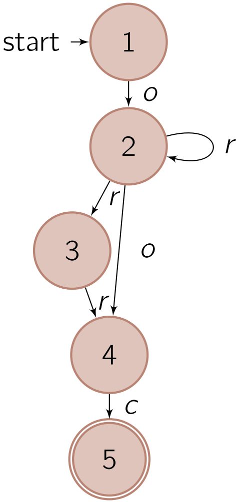

Abbildung: Graphical Representation of File Usage System C

Verifying Programs

$$C \otimes F = \begin{pmatrix} \cdot & o & \cdot & \cdot & \cdot \\ \cdot & r & r & o & \cdot \\ \cdot & \cdot & \cdot & r & \cdot \\ \cdot & \cdot & \cdot & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \otimes \begin{pmatrix} \cdot & o \\ c & r \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & o & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & o & \cdot & \cdot \\ \cdot & \cdot & \cdot & r & \cdot & r & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Verifying Programs

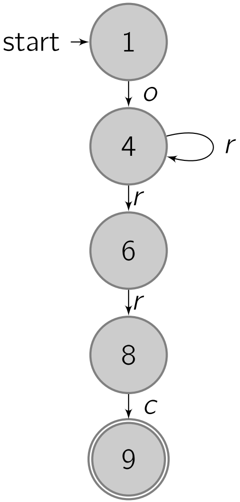

Abbildung: Graphical Representation of $C \otimes F$

Verifying Programs

Isomorphic?

Verifying Programs

Isomorphic?

General Statement

1. Assume we pick a path on program P's side, that complies to the usage scenario U. Then a corresponding path will be present in $P \otimes U$.
2. Assume we pick a path on program P's side, that does not comply completely to the usage scenario U. Then a "corresponding" path in $P \otimes U$ will end as soon as the path does not comply to U. This will result in $P \not\simeq P \otimes U$.

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Thus we can state:

Theorem

Given P, a control flow graph (CFG) of a program, and a usage scenario U, program P complies to U if and only if $P \simeq P \otimes U$.

The Kronecker Sum

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- programs will use several objects (classes, modules, packages, . . .) for implementing their own task.

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- checking against each of their usage scenarios could be done one after the other.
- here: check a program against two and more usage scenarios at the same time.
- Kronecker sum.

The Kronecker Sum

Assume that $U \in M_n(\mathcal{U})$ and $W \in M_m(\mathcal{W})$ and that $\mathcal{U} \cap \mathcal{W} = \emptyset$.

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Introduce self loops on both sides.

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Introduce self loops on both sides.

Definition

$$U \oplus W = U \otimes \left(W + \left(\sum_{x \in \mathcal{U}} x \right) I_m \right) + \left(U + \left(\sum_{y \in \mathcal{W}} y \right) I_n \right) \otimes W =$$

$$U \otimes W + U \otimes \left(\left(\sum_{x \in \mathcal{U}} x \right) I_m \right) + U \otimes W + \left(\left(\sum_{y \in \mathcal{W}} y \right) I_n \right) \otimes W.$$

Note that $U \otimes W = Z_{n \cdot m}$ because the edge labels of U and W are disjoint. Hence we get

$$U \oplus W = U \otimes I_m + I_n \otimes W.$$

The Kronecker Sum

Theorem

With the definitions above, a program A can be checked against two usage scenarios U and W by calculating

$$A \odot (U \oplus W) = A \odot (U \otimes I_m + I_n \otimes W)\,.$$

The Kronecker Sum

We can even check a program against more than two usage scenarios at the same time.

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$$\bigoplus_{x \in \mathcal{X}} x$$

similar to the sigma notation for standard sums.

The Kronecker Sum

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$$\bigoplus_{x \in \mathcal{X}} x$$

similar to the sigma notation for standard sums.

Theorem

With the definitions above, a program A can be checked against usage scenarios U_i where $i = 1, \ldots, k$ by calculating

$$A \odot \left(\bigoplus_{i=1}^{k} U_i \right).$$

The Kronecker Sum

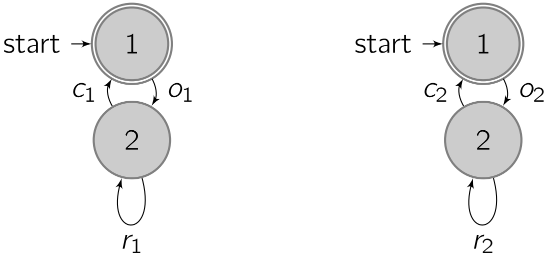

Abbildung: Graphical Representation of File Usage Scenarios U and W

The Kronecker Sum

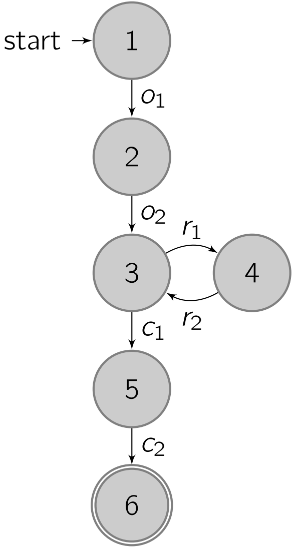

Abbildung: CFG of Program A

The Kronecker Sum

$$U = \begin{pmatrix} \cdot & o_1 \\ c_1 & r_1 \end{pmatrix},$$
$$W = \begin{pmatrix} \cdot & o_2 \\ c_2 & r_2 \end{pmatrix},$$

and

$$A = \begin{pmatrix} \cdot & o_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & o_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & r_1 & c_1 & \cdot \\ \cdot & \cdot & r_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The Kronecker Sum

$$
\begin{aligned}
U \oplus W =& U \otimes I_2 + I_2 \otimes W = \\
& \begin{pmatrix} \cdot & o_1 \\ c_1 & r_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix} + \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix} \otimes \begin{pmatrix} \cdot & o_2 \\ c_2 & r_2 \end{pmatrix} = \\
& \begin{pmatrix} \cdot & \cdot & o_1 & \cdot \\ \cdot & \cdot & \cdot & o_1 \\ c_1 & \cdot & r_1 & \cdot \\ \cdot & c_1 & \cdot & r_1 \end{pmatrix} + \begin{pmatrix} \cdot & o_2 & \cdot & \cdot \\ c_2 & r_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & o_2 \\ \cdot & \cdot & c_2 & r_2 \end{pmatrix} = \\
& \begin{pmatrix} \cdot & o_2 & o_1 & \cdot \\ c_2 & r_2 & \cdot & o_1 \\ c_1 & \cdot & r_1 & o_2 \\ \cdot & c_1 & c_2 & r_1 + r_2 \end{pmatrix} .
\end{aligned}
$$

The Kronecker Sum

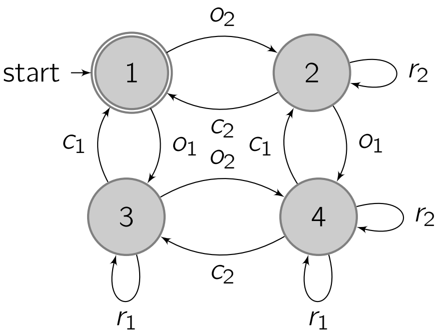

Abbildung: Graphical Representation of $U \oplus W$

The Kronecker Sum

$A \odot (U \oplus W) =$

$$\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & o_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & o_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & o_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & o_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r_1 & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r_1 & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}.$$

The Kronecker Sum

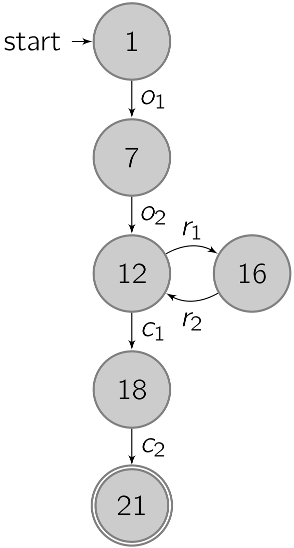

Abbildung: Graphical Representation of $A \odot (U \oplus W)$

The Kronecker Sum

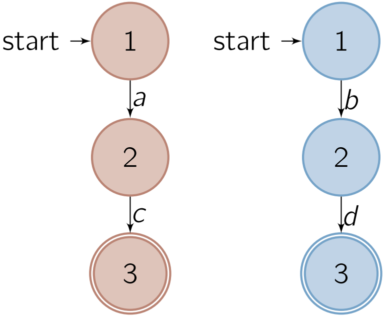

Abbildung: Our introductory example from a previous section

Kronecker Sum

Matrices

$$A = \begin{pmatrix} \cdot & a & \cdot \\ \cdot & \cdot & c \\ \cdot & \cdot & \cdot \end{pmatrix}$$

and

$$B = \begin{pmatrix} \cdot & b & \cdot \\ \cdot & \cdot & d \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Kronecker Sum

Kronecker Product 1

$$A \otimes I_3 = \begin{pmatrix} \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Kronecker Sum

Kronecker Product 2

$$I_3 \otimes B = \begin{pmatrix} \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} .$$

Kronecker Sum

Kronecker Sum

$$A \otimes I_3 + I_3 \otimes B = \begin{pmatrix} \cdot & b & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & d & \cdot & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & b & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & d & \cdot & c & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The Kronecker Sum

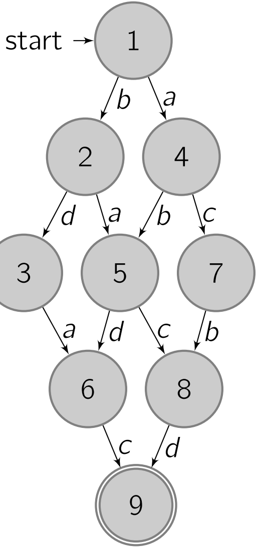

Abbildung: Resulting Interleavings Graph

The Kronecker Sum

For more on Kronecker Algebra and its many applications see `https://kronalg.blieberger.at`.