3. ÜBUNG
104.283 Diskrete Mathematik für Informatik

(19) Show that all bases of a matroid $M = (E, S)$ have the same cardinality.
Lösung: *Suppose that there are two bases A and B of different cardinality, $|B| < |A|$. Since bases are independent sets, by the exchange axiom there is an element $b \in B \setminus A$ such that $A \cup \{b\}$ is an independent set. Since a basis is a maximal independent set, and A was assumed to be a basis, this is a contradiction.*

(20) Let $G = (V, E)$ be an undirected graph. Set $M_k(G) = (E, S)$ where

$$S = \{A \subseteq E : A = F \cup M, F \text{ acyclic and } |M| \leq k\}.$$

Show that $M_k(G)$ is a matroid!
Lösung: *The empty set is in S and a subset of a set in S is again in S. We need to show that the exchange axiom holds: let A and B be in S with $|B| = |A| + 1$, then there is an edge $e \in B \setminus A$ such that $A \cup e$ is in S.*

Choose F_A such that M_A is minimal. If $|M_A| < k$ we can add any edge from $B \setminus A$ to A. Otherwise, we have to show that there must be an edge in B which connects two components of F_A. Let $T_1, \ldots, T_c$ be the components of F_A. Then the restriction of the forest F_B to any of the vertex sets $V(T_i)$ is still a forest and has at most $V(T_i) - 1$ edges. Therefore

$$\sum_i |E(B|_{V(T_i)})| \leq \sum_i (V(T_i) - 1) + k = |A|.$$

(21) One can show (but this is not part of this exercise) that a matroid is characterised by listing the set of its bases.
(a) Use this fact to list all matroids with precisely 3 elements.
(b) Find two non-isomorphic graphs with three edges corresponding to the same matroid.
(c) Find a matroid, which does not correspond to a graph.
(d) Find a set E and a set $\mathcal{B}$ of subsets of E which all have the same cardinality, such that $\mathcal{B}$ is not the set of bases of a matroid.

Lösung:
- *Separating bases within a matroid with bars and matroids with by commas:*
  (a) $\emptyset$
  (b) 1, 2, 3, 1|2, 1|3, 2|3, 1|2|3
  (c) 12, 13, 23, 12|13, 12|23, 13|23, 12|13|23
  (d) 123
- *The two non-isomorphic trees on four vertices (the path and the star) both have only one base which consists of all elements of the ground set.*
- *The matroid with a four-element groundset and all two-element subsets being bases cannot correspond to a graph. To see this, one has to show that there is no graph with six edges, all pairs being spanning trees.*
- *Part a of this exercise tell us that taking subsets of the same cardinality of a three element set always yields a set of bases. Thus, we take $E = \{1, 2, 3, 4\}$. We guess that 12|34 is not a set of bases, and show that this is the case because it violates the exchange axiom with $B = 12$ and $A = 3$.*

(22) Let J be the set of jobs $\{0, 1, 2, 3, 4\}$ and W be the set of workers $\{04, 0, 0123, 12\}$. Suppose that a job can be done if its number appears in the 'name' of the worker.

List all maximal sets of jobs that can be done simultaneously, i.e., the bases of the matroid considered in the lecture. Then use the greedy algorithm to find an optimal job assignment, where the priority of a job is given by its number.

(23) Let V be a vector space and E be a finite subset of V. Let I be the set of linearly independent subsets of E. Show that (E, I) is a matroid.

(24) Compute the distances from vertex v_0 in the graph of exercise 17 using Dijkstra's algorithm, highlighting the distance tree computed by Dijkstra's algorithm with extra colour. Show, using a very small counterexample, that the distance tree (as produced for example by Dijkstra's algorithm) is, in general, not a minimum spanning tree, and that the minimum spanning tree is in general not the tree with minimal distances.

(25) Give an example of a weighted graph G for which Dijkstra's algorithm fails.

(26) Compute the distances from vertex v_0 in the graph below using the algorithm by Bellman, Ford and Moore. Explain why this implies that we cannot define a 'distance tree' for graphs with negative edge weights.