Computernumerik (Visual Computing) Test 1

22.November 2024

Time: 100 minutes

I tried to copy the exact test questions as good as possible. The given solutions are just my personal solutions, so no guarantee they are 100% correct.

1.1

Given are the knots $x_0 = -1, x_1 = 0, x_2 = 1$ and values $f_0 = 2, f_1 = 0, f_2 = 0$

a) Calculate the Lagrange polynomials l_i and the resulting quadratic interpolation polynomial p(x).

$l_0 = \frac{x-0}{-1-0} \cdot \frac{x-1}{-1-1} = \frac{x^2-x}{2}$
$l_1 = \frac{x+1}{0+1} \cdot \frac{x-1}{0-1} = \frac{x^2-1}{-1} = 1 - x^2$
$l_2 = \frac{x+1}{1+1} \cdot \frac{x-0}{1-0} = \frac{x^2+x}{2}$

$p(x) = \sum_{i=0}^{n} f(x_i) l_i(x) = 2(\frac{x^2-x}{2}) = x^2 - x$

b) Assume the form $p(x) = \alpha + \beta x + \gamma x^2$. Let vector $x = (\alpha, \beta, \gamma)^T$. Create a system of linear equations in form of the matrix A and vector b, so that $Ax = b$. You do not have to solve the system. Use x_o, x_1, x_2 and f_0, f_1, f_2 as stated in the beginning.

$$\begin{pmatrix} 1 & \mathrm{x}_0 & \mathrm{x}_0^2 \\ 1 & \mathrm{x}_1 & \mathrm{x}_1^2 \\ 1 & \mathrm{x}_2 & \mathrm{x}_2^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha + \beta x_0 + \gamma x_0^2 \\ \alpha + \beta x_1 + \gamma x_1^2 \\ \alpha + \beta x_2 + \gamma x_2^2 \end{pmatrix} = \begin{pmatrix} p(x_0) \\ p(x_1) \\ p(x_2) \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}$$

$$\Longrightarrow A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

c) Explain why the Chebyshev knots are a good choice. What is the growth of the Lebesgue constant in terms of n when using Chebyshev knots.

When sampling uniformly, the biggest errors occur near the interval borders. To counter that, the Chebyshev knots have a higher sampling rate near the interval borders.

$A_n^{Cheb} \leq \frac{2}{\pi} ln(n+1) + 1 \rightarrow$ it grows logarithmically

1.2

a) Estimate a bound for $||f - p||_{\infty,[a,b]}$ in terms of $f^{(n+1)}$.

script WS23/24, page 15

$$||f - p||_{\infty,[a,b]} \leq ||\omega_{n+1}||_{\infty,[a,b]} \frac{||f^{(n+1)}||_{\infty,[a,b]}}{(n+1)!} \leq (b-a)^{n+1} \frac{||f^{(n+1)}||_{\infty,[a,b]}}{(n+1)!}$$

b) Define the Lebesgue constant in terms of the knots $x_0, ...x_n$.

$$\mathrm{A}_n = \underset{x \in [-1,1]}{max} \sum_{i=0}^{n} |l_i(x)|$$

where l_i is the Lagrange polynomial of x_i

c) Bound the error $||f - I_n f||_{\infty,[-1,1]}$ in terms of the Lebesgue constant.

script WS23/24, page 23

$$||f - I_n f||_{\infty,[-1,1]} \leq (1 + A_n) \underset{q \in P_n}{min} ||f - q||_{\infty,[-1,1]}$$

1.3

Given is the Quadrature formula $Q(f) = \frac{1}{2} f(\frac{1}{4}) + \frac{1}{2} f(\frac{3}{4})$

a) Is Q(f) exact for $f(x) = x^2 + 1$ on the interval [0,2]?

$\int_0^2 x^2 + 1 dx = \frac{4}{3} = \frac{64}{48}$

$Q(f) = \frac{1}{2} \cdot \frac{17}{16} + \frac{1}{2} \cdot \frac{25}{16} = \frac{21}{16} = \frac{63}{48}$

$\longrightarrow error = \frac{1}{48}$

b) What is the degree of exactness of Q(f)? Justify your answer.

We will check if Q(f) is exact for a basis of P_n on the interval [0,1].

P_o: $f(x) = 1, \int_0^1 1dx = 1, Q(f) = 1$ ✓

P_1: $f(x) = x, \int_0^1 xdx = \frac{1}{2}, Q(f) = \frac{1}{2}$ ✓

As seen in a), Q(f) ist not exact for all $p \in P_2$. Thus, the degree of exactness is 1.

c) What is the degree of exactness for Gaussian quadrature when using 2 interpolation points? What is the general degree of exactness for Gaussian quadrature using $n+1$ interpolation points?

$n+1$ points $\rightarrow$ degree $2n+1$

2 points $\rightarrow$ degree $2(1)+1=3$

1.4

$\phi(x) = \sqrt{2x+1} - 1, x > 0$

Estimate an upper bound on the relative condition number of $\phi(x)$. Would you consider the problem well-conditioned? What issues could arise when x is near 0? Give a stable implementation of $\phi(x)$.

$\phi'(x) = 2 \cdot \frac{1}{2}(2x+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x+1}}$

$\kappa_{rel}(x) = \frac{|\phi'(x)|}{|\phi(x)|}|x| = |\frac{1}{\sqrt{2x+1}}| \cdot \frac{1}{|\sqrt{2x+1}-1|} \cdot |x| \overset{x\geq 0}{=} \frac{1}{\sqrt{2x+1}} \cdot \frac{1}{\sqrt{2x+1}-1} \cdot x$

$= \frac{1}{\sqrt{2x+1}} \cdot \frac{(\sqrt{2x+1}+1)}{(\sqrt{2x+1}-1)(\sqrt{2x+1}+1)} \cdot x$

$= \frac{1}{\sqrt{2x+1}} \cdot \frac{\sqrt{2x+1}+1}{2x} \cdot x = \frac{\sqrt{2x+1}+1}{2\sqrt{2x+1}} = \frac{1}{2} \cdot (\frac{\sqrt{2x+1}}{\sqrt{2x+1}} + \frac{1}{\sqrt{2x+1}})$

$= \frac{1}{2}(1 + \underbrace{\frac{1}{\sqrt{2x+1}}}_{\leq 1}) \leq \frac{1}{2}(1+1) = 1$

The problem is well-conditioned because $\kappa_r el \leq 1$ for all $x > 0$.
Near 0 the problem arises, that we subtract 2 very similar numbers from each other, which is ill-conditioned. A stable implementation would be:

$\phi(x) = \sqrt{2x+1} - 1 = \frac{(\sqrt{2x+1}-1)\cdot(\sqrt{2x+1}+1)}{\sqrt{2x+1}+1} = \frac{(\sqrt{2x+1}-1)\cdot(\sqrt{2x+1}+1)}{\sqrt{2x+1}+1} = \frac{2x}{\sqrt{2x+1}+1}$

Since $\sqrt{\cdot}, \div, \cdot, +$ are all well-conditioned.

1.5

Decide if True or False.

a)

1. For a function f and knots $x_0, ..., x_n$ there exists a unique interpolation polynomial $p \in P_n$ that interpolates f in $x_0, ..., x_n$. True
2. More interpolation knots always mean a lower error. False
3. For Lagrange-interpolation with uniformly sampled knots, the Lebesgue constant grows with O(log n). False (growth is exponential, see script WS23/24, page 24)

b)

1. The cost for evaluating a polynomial $p \in P_n$ with the Neville Scheme is $O(n^2)$. True
2. Taking the square root of a positive number is well-conditioned. True
3. If a problem is well conditioned, then an algorithm that realizes it is always stable. False

c)

1. Quadrate rules with $\sum \omega_i = 1$ on the interval [0,1] are always correct for constant functions. True
2. The compostite Simpson rule converges with order 3. False (Order 4, script WS23/24, page 47)
3. The weights for Gaussian quadrature are always positive. True

d) Fill in the blanks:

1. The composite trapezoidal rule is of order <u>(2)</u>.
2. The simpson rule has a degree of exactness of <u>(3)</u>.
3. Subtraction of two similar number is <u>(ill)</u> conditioned.