

Exercise 4

Discrete Mathematics

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November 5, 2020

Exercise 31

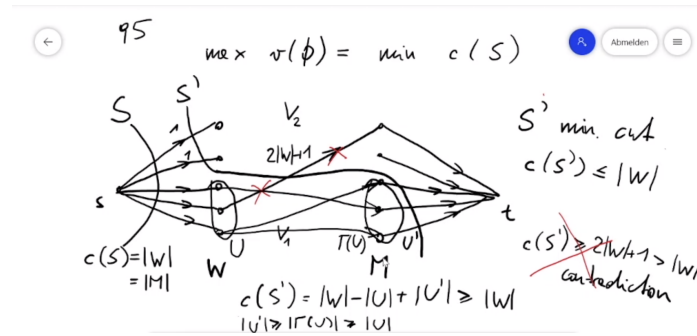
Note from TUWEL: Assume $|V|_1 = |V|_2$ and (this I haven't seen in time) use $W = V_1 \setminus S$ instead of the W in the task description. The new W led to a lot of discussion in the exercise presentation.

\Leftarrow If there is a perfect matching in G then there must be $|W| \leq \mathcal{N}(W)$, because the edges of the perfect matching match each vertex in W to a distinct vertex in $\mathcal{N}(W)$, and this is impossible if $\mathcal{N}(W) < |W|$.

\Rightarrow The capacity of all edges that were originally in G is infinite. Therefore, any minimal cut cannot contain half of any such edges. Hence, we have $\mathcal{N}(W) \subseteq S \cap V_2$. but this means that (second line is equivalent to second bullet point which is to show)

$$\begin{aligned}
 c(S, \bar{S}) &= \sum_{x \in S, y \in \bar{S}} c(x, y) \\
 &= \sum_{x \in (S \cap \{s\}), y \in (\bar{S} \cap V_1)} c(x, y) + \sum_{x \in (S \cap V_2), y \in (\bar{S} \cap \{t\})} c(x, y) \\
 &\leq n - |W| + |\mathcal{N}(W)| \\
 &\leq n - |W| + |W| \\
 &= n
 \end{aligned}$$

Any (minimal) cut S has to go through at least $n = |V|_1 = |V|_2$ vertices of weight 1. Therefore, $c(S) \geq n$. So any minimal cut has capacity n ; therefore, there is a flow with value n . Such a flow sends one unit to each vertex in V_1 , and sends one unit from each vertex in V_2 to t ; therefore, the edges marked 1 in G by such a flow form a perfect matching of G 's vertices.

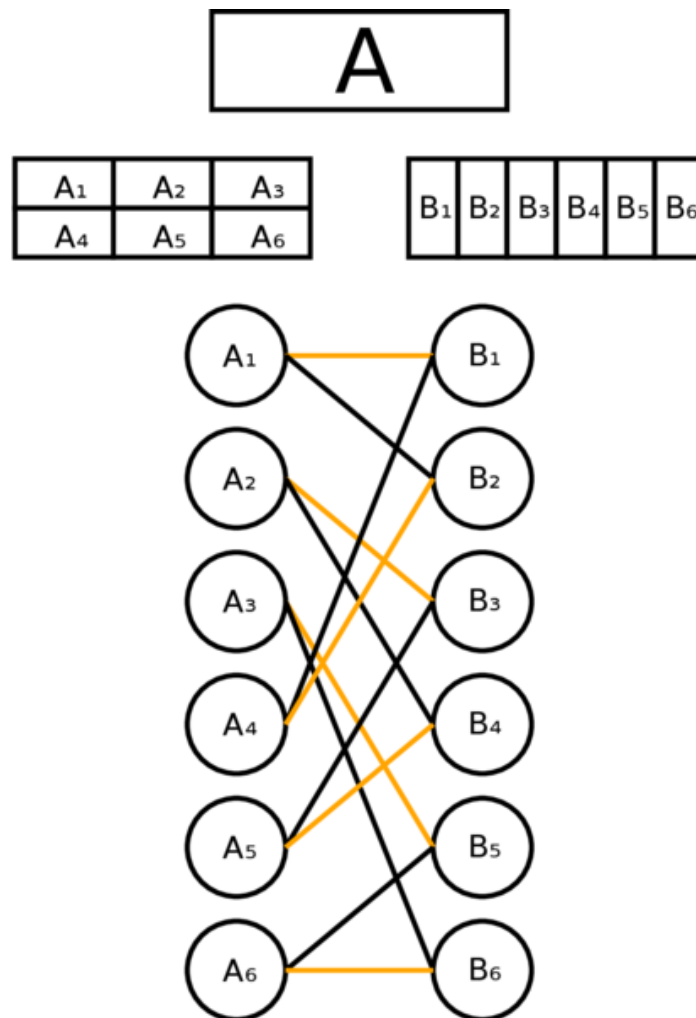


Exercise 32

Vowi

For every area A_1, A_2, \dots, A_m and B_1, B_2, \dots, B_m we draw a node. For every set A_i and B_j that share a common area $A_i \cap B_j \neq \emptyset$ we draw an edge between their nodes. This creates a bipartite graph. To prove the existence of such a permutation π we need to find a perfect matching.

Example:



Let S be a subset of the vertices of A_1, A_2, \dots, A_m of size s . Then S represents an area of size $\frac{sa}{m}$. Now you need at least s parts of B_1, B_2, \dots, B_m to cover this area, so $|\mathcal{N}(S)| \geq |S|$. By Hall's Marriage Theorem, we now have a perfect matching.

Exercise 33

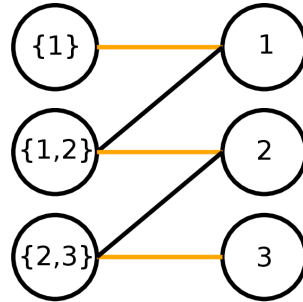
vowi

We construct a bipartite graph by placing a node for every set B_i on the left side. Then we place a node for every element a_j on the right side. We connect nodes B_i with every node a_j which contains an element of B_i . The task of finding an injective mapping is now equivalent to finding a perfect matching in the graph.

Example:

$$A = \{a_1, \dots, a_n\} = \{1, 2, 3\}$$

$$B = \{B_1, \dots, B_n\} = \{\{1\}, \{1, 2\}, \{2, 3\}\} \subseteq 2^A$$



By Hall's marriage theorem there is a perfect matching in the constructed graph if and only if for every subset W of vertices of the right side $|W| \leq |\mathcal{N}(W)|$ holds. By our reduction, this proves that the injective mapping exists if and only if for all $I \subseteq \{1, 2, \dots, n\}$ it holds that $|\cup_{i \in I} B_i| \geq |I|$.

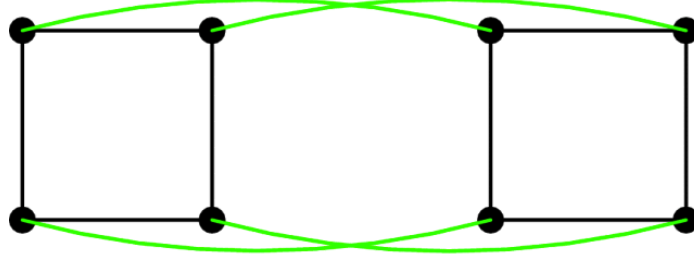
Exercise 34

Proof by induction.

Base: The hypercube Q_1 is the complete graph on two vertices. The only edge is incident to all vertices. Thus, this is a perfect matching.

Induction step: Assume Q_{n-1} has a perfect matching. We know that hypercubes Q_n are constructed from the disjoint union of two hypercubes Q_{n-1} by adding an edge from each vertex in one Q_{n-1} hypercube to the single "corresponding" (the "same" so to say) vertex in the other Q_{n-1} hypercube.

Example: Construction of Q_3 by connecting pairs of corresponding vertices in two copies of Q_2 .



Those added edges form a matching, as there is exactly one corresponding vertex for each vertex. The matching is also complete, as every vertex is incident to one such edge. Thus, we have a perfect matching for Q_n . This completes our proof.

Exercise 35

\Rightarrow Assume $G = (V, E)$ is bipartite. That means $V = A \cup B$ such that $A \cap B = \emptyset$ and that all edges $e \in E$ are such that e is of the form $\{a, b\}$ where $a \in A$ and $b \in B$ (definition bipartite graph).

Suppose G has (at least) one odd cycle C of length n . Let $C = (v_1, v_2, \dots, v_n, v_1)$. Wlog let $v_1 \in A$. It follows that $v_2 \in B$, then $v_3 \in A$ and so on.

Hence we see that $\forall k \in \{1, 2, \dots, n\}$, we have:

$$v_k \in \begin{cases} A : & k \text{ odd} \\ B : & k \text{ even} \end{cases}$$

But as n is odd, $v_n \in A$.

But $v_1 \in A$, and $v_n, v_1 \in C_n$.

So $v_n, v_1 \in E$ which contradicts the assumption that G is bipartite.

Hence if G is bipartite, it has no odd cycles.

\Leftarrow It is enough to consider G as being connected, as otherwise we could consider each component separately.

Suppose G has no odd cycles.

Choose any vertex $v \in G$.

Divide G into two sets of vertices like this:

Let A be the set of vertices such that the shortest path from each element of A to v is of odd length; Let B be the set of vertices such that the shortest path from each element of B to v is of even length.

Then $v \in B$ and $A \cap B = \emptyset$.

Suppose $a_1, a_2 \in A$ are adjacent.

Then there would be a closed walk of odd length $(v, \dots, a_1, a_2, \dots, v)$.

A graph containing a closed walk of odd length also contains an odd cycle. It follows that G would then contain an odd cycle.

This contradicts our initial supposition that G contains no odd cycles.

So no two vertices in A can be adjacent.

By the same argument, neither can any two vertices in B be adjacent.

Thus A and B satisfy the conditions for $G = (A \cup B, E)$ to be bipartite.

Exercise 36

(a) As shown by Chartrand ¹

Proposition 1. Every eulerian graph is sequential.

Proof. If G is an eulerian graph, then G contains a closed path P containing each line of G exactly once, say $P : x_0, x_1, \dots, x_{q-1}, x_q = x_0$, where x_i and x_{i+1} are adjacent for $i = 0, 1, \dots, q-1$. This ordering of the edges of G serves to show that G is sequential.

Proposition 2. Every hamiltonian graph is sequential.

Proof. Let C be a hamiltonian cycle of a hamiltonian graph G whose points are arranged cyclically as, say, $v_0, v_1, \dots, v_{p-1}, v_p = v_0$. To show that G is sequential, we exhibit an appropriate ordering of the edges of G . We begin the sequence of lines by selecting all those diagonals incident with v_0 (there may be none). These lines may be taken in any order, and, each two are adjacent with each other. We follow these with the edge v_0v_1 . The next edges in the sequence are those diagonals incident with v_1 (again, there may be none). As before, these lines may be taken in any order. The next line in the sequence is v_1v_2 , followed by all those diagonals incident with v_2 which are not in the part of the sequence already formed. We continue this until we finally arrive at the edge $v_{p-1}v_p = v_{p-1}v_0$, which is adjacent with the first line in the sequence. From the way the sequence was produced, makes that every line of G appears exactly once and that any two consecutive edges in the sequence are adjacent as are the first and last lines. Thus G is sequential.

Theorem 1. A necessary and sufficient condition that the line-graph \bar{G} of a graph G be hamiltonian is that G is sequential.

Proof. The result follows by observing that the points of \bar{G} can be ordered $v_0, v_1, \dots, v_{p-1}, v_p = v_0$, where v_i and v_{i+1} are adjacent for $i = 0, 1, \dots, p-1$ if and only if \bar{G} is hamiltonian, and such an ordering is possible if and only if the edges of G can be ordered $x_0, x_1, \dots, x_{p-1}, x_p = x_0$, where x_i and x_{i+1} are adjacent for $i = 0, 1, \dots, p-1$. This latter condition states that G is sequential.

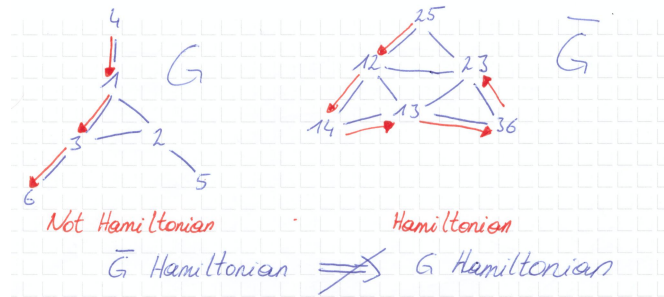
Proposition 1 and 2 and Theorem 1 yield the required corollaries:

Corollary 1: If G is an eulerian graph, then \bar{G} is both eulerian and hamiltonian.

Corollary 2: If G is a hamiltonian graph, then \bar{G} is hamiltonian.

From the fact that \bar{G} is Hamiltonian, we cannot conclude that G is Hamiltonian.

¹Gary Chartrand. On Hamiltonian Line-Graphs. 1968.



- (b) Assume G is an Eulerian graph (connected, different components are irrelevant for subdivision). Then all vertices of G have even degree. A subdivision G' of G replaces each edge (v_n, v_m) by two (or more, but this would be analog reasoning) edges $(v_n, v_x), (v_x, v_m)$. The degree of v_n, v_m has not changed. The degree of v_x is two. All vertex degrees are even. Therefore, G' has a Eulerian tour.

Assume G is an Hamiltonian graph (connected, different components are irrelevant for subdivision). Then there is an Hamiltonian cycle in it. Then every vertex v is visited exactly once (except start=end). v has a successor v' in the cycle. A subdivision of G' replaces the edge (v, v') with two (for simplicity; could be more, but the idea still holds) edges $(v, u), (u, v')$. Now u is the success or of v and v' is the successor of u . Consequently, each vertex is still visited exactly once. Therefore, G' is a Hamiltonian graph.

Exercise 37

We proof that $K_{m,n}$ has a hamiltonian cycle if and only if $m = n$.

\Rightarrow Assume $K_{m,n}$ has a hamiltonian cycle. As $K_{m,n}$ is bipartite, the cycle visits the two subsets alternately. The cycle also visits every vertex. Assume $m \neq n$ then there are two vertices from the same subset/side that must be connected. This contradicts the bipartite condition. Therefore, we get $m = n$ if $K_{m,n}$ has a hamiltonian cycle.

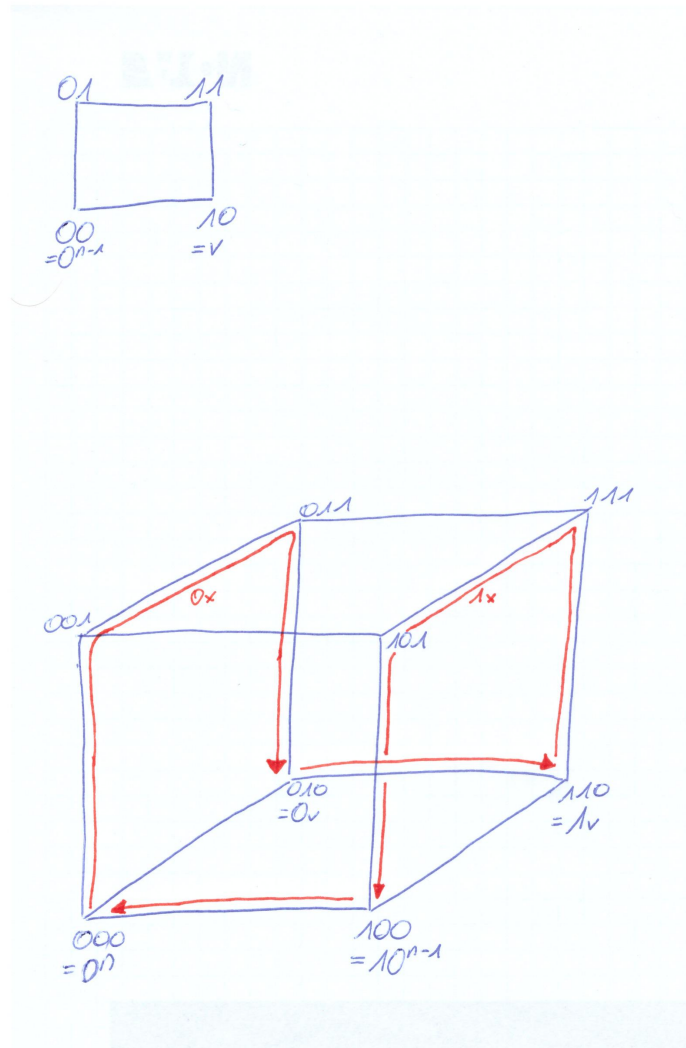
\Leftarrow Assume $m = n$. $K_{m,n}$ is complete and bipartite. Then, for subsets X, Y there is a path $x_0, y_0, x_1, y_1, x_2, y_2, \dots, x_n, y_n$. This path visits every vertex exactly once. Connecting x_0, y_n gives a cycle. Therefore, if $m = n$ then there is a hamiltonian cycle.

Exercise 38

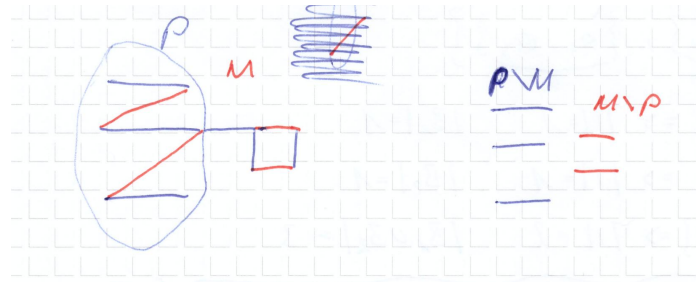
Proof by induction on n . In the base case $n = 2$, the 2-dimensional hypercube, the length four cycle starts from 00, goes through 01, 11, and 10, and returns to 00.

Suppose now that every $(n - 1)$ -dimensional hypercube has an Hamiltonian cycle. Let $v \in \{0, 1\}^{n-1}$ (space of all $(n - 1)$ -length vectors consisting of 0s and 1s) be a

vertex adjacent to 0^{n-1} (the notation 0^{n-1} means a sequence of $n - 1$ zeroes) in the Hamiltonian cycle in a $(n - 1)$ -dimensional hypercube. The following is a Hamiltonian cycle in an n -dimensional hypercube: have a path that goes from 0^n to $0v$ by passing through all vertices of the form $0x$ (this is simply a copy of the Hamiltonian path in dimension $(n - 1)$, minus the edge from v to 0^{n-1}), then an edge from $0v$ to $1v$, then a path from $1v$ to 10^{n-1} that passes through all vertices of the form $1x$, and finally an edge from 10^{n-1} to 0^n . This completes the proof of the Theorem.



Exercise 39



We know that every second edge of P is in M . We denote this set as X . As P is extending, it starts and ends with an edge, which does not belong to the matching M . Therefore, the number of edges in P must be odd. Consequently $|P \setminus M| = |X| + 1$

Let $v \in P$ be arbitrary. By definition of matching, v is incident to exactly one edge $x \in X$. P is alternating. Therefore, v is also incident to exactly one edge $x' \in P \setminus X = P \setminus M$. Hence, $P \setminus X = P \setminus M$ is a matching of P . Additionally, this shows that no edge in $E \setminus P$ that is incident to v can be part of the matching M . As v was chosen arbitrarily and the matching $M \setminus P$ (everything outside of P so to say) was left unchanged, we get that $M \Delta P$ is a matching, too.

As the matching $M \setminus P$ was left unchanged, especially its cardinality has not changed either. With our initial cardinality calculation of $|P \setminus M|$ we get $|M \Delta P| = |M| + 1$.

Exercise 40

$$\chi(K_n) = n \implies \chi(H) = n \leq \chi(G)$$

In general, there is no bound on the chromatic number of a graph in terms of the size of its largest complete subgraph, since there are graphs containing no triangle, but having arbitrarily large chromatic numbers. ²

Any odd cycle C_{2k+1} is a graph with $\chi(G) = 3$ and does not admit K_3 as subgraph.

²S. Wagon. A Bound on the Chromatic Number of Graphs without Certain Induced Subgraphs. 1978.