TU Wien
Institute of Logic and Computation
Algorithms and Complexity Group

186.814 Algorithmics VU 6.0

Exam (winter term 2021/22)

January 28, 2022

Write your answers on blank paper (with your name and matriculation number on the top of each page), which you finally scan/take pictures of and submit as one PDF file via TUWEL.

For further details we refer to the earlier distributed organizational instructions.

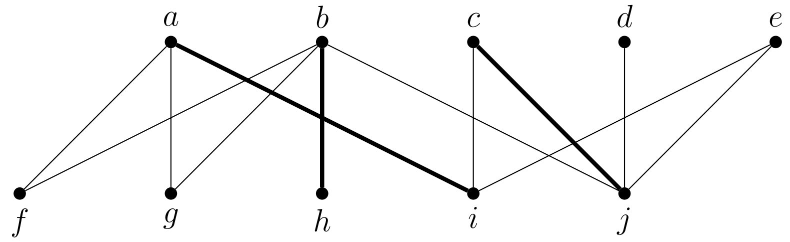
Question	Points	(max)	Question	Points	(max)
1		(10)	4		(10)
2		(10)	5		(10)
3		(5)	6		(5)

Total: (50)

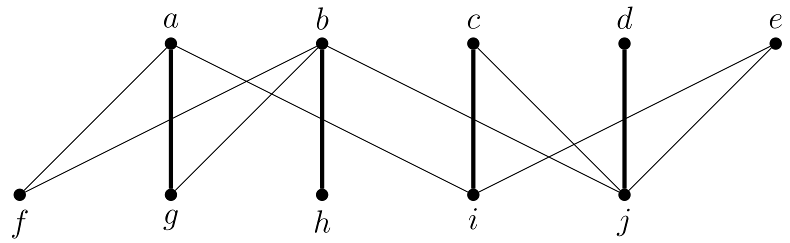
Good luck and best of success!

Question 1: Matchings in Bipartite Graphs **(10 Points)**

Consider the following bipartite graph $G = (V_1 \cup V_2, E)$ where $V_1 = \{a, b, c, d, e\}$ and $V_2 = \{f, g, h, i, j\}$, with a matching M indicated by bold lines.



Here is the same graph with a maximum matching M' indicated by bold lines.



a) (2 Points) Specify an M-augmenting path P in G such that M is the symmetric difference of $E(P)$ and M'.

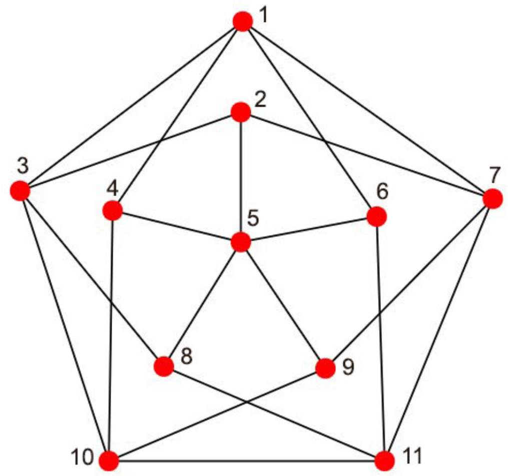
b) (3 Points) Specify the one-sided canonical decomposition (Red, Green) of G with respect to M by listing the vertices in Red.

c) (3 Points) Specify the minimum vertex cover C of G that you obtain from the one-sided canonical decomposition by listing the vertices that are in C. Argue why G has no smaller vertex cover.

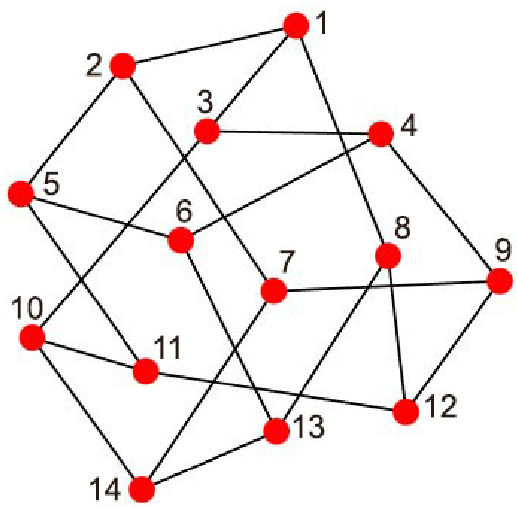
d) (2 Points) Specify a subset $S_1 \subseteq V_1$ such that $|N(S_1)| < |S_1|$, and a subset $S_2 \subseteq V_2$ such that $|N(S_2)| < |S_2|$.

Question 2: Planar Graphs, A* Algorithm (10 Points)

a) (3 Points) Prove in one or few sentences that the following graph is not planar and mark respective nodes/edges used in the proof in the graph.

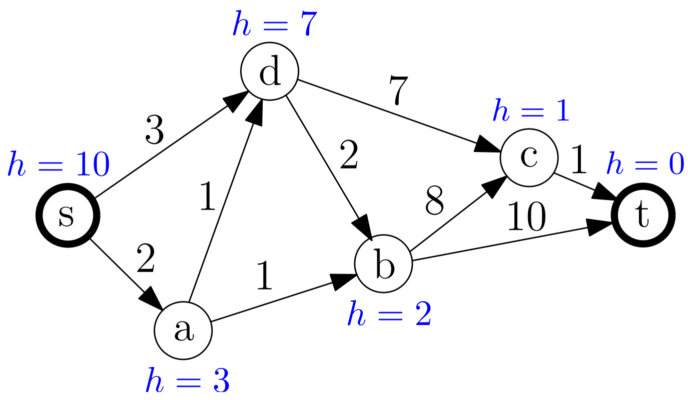


b) (3 Points) Show that the following graph contains a subdivision of $K_{3,3}$. Mark the nodes of the two partitions in different ways and all edges belonging to the subdivision.



c) (4 Points)

Perform the A* algorithm on the following graph from start node s to target node t. List the nodes in an order in which they might be first reached by the algorithm and additionally in the order in which they are expanded. (2 points)



Are the heuristic values given for the nodes in the above graph admissible? Give an explanation why or why not. (2 points)

Question 3: Fixed Parameter Tractability **(5 Points)**

Give an FPT algorithm for the following problem and specify its running time in $\mathcal{O}$-notation.

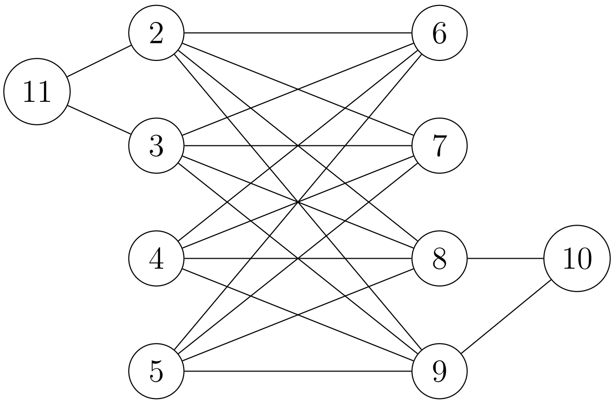
Instance: A graph G, a number $k \in \mathbb{R}$, a function $w\colon V(G) \to [1, \infty)$ which assigns each vertex $v \in V(G)$ a weight $w(v) \in \mathbb{R}$ with $w(v) \geq 1$. *Parameter*: k. *Question*: Is there a set $S \subseteq V(G)$ with $\sum_{v \in S} w(v) \leq k$ such that for every edge $uv \in E(G)$ we have that $u \in S$ or $v \in S$?

Question 4: Structural Decompositions and Algorithms **(10 Points)**

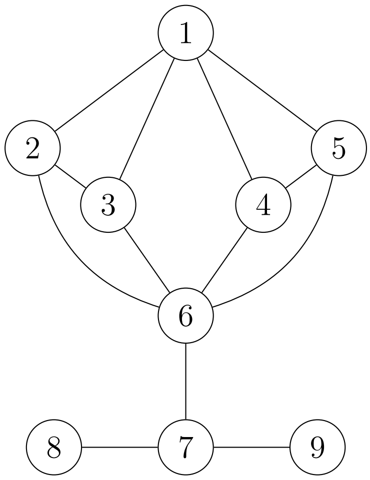
a) (4 Points)
The *star graph* S_n of order n is a tree on n vertices with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1.

Given a graph G, we want to compute the largest n such that G contains S_n as an *induced* subgraph. Use FPT results for MSO model checking to prove that this problem is FPT parameterized by the treewidth of the input graph G.

b) (3 points)
Prove that the graph depicted below has treewidth at least 4.



c) (3 points)
Prove that the graph shown below has treewidth at most 3.



Question 5: (Integer) Linear Programming **(10 Points)**

Consider the following Binary Integer Program (BIP):

$$\max \sum_{i=1}^{4} c_i x_i \tag{1}$$

$$\sum_{i=1}^{4} a_i x_i \leq b \tag{2}$$

$$x_i \in \{0,1\} \qquad \forall i \in \{1,2,3,4\} \tag{3}$$

a) (1 Point) Formulate the LP relaxation of BIP.

b) (2 Points) Assume you found an optimal solution x^* for the LP relaxation of BIP and x^* is feasible for BIP. Explain why x^* is also an optimal solution for BIP.

For the subsequent tasks consider the following clustering problem. Given is a weighted undirected simple graph $G = (V, E)$ with nodes $V = \{1, \ldots, n\}$, edges $E \subseteq V \times V$ and costs $c(e) > 0$ associated with each edge $e \in E$. The set $E_i \subseteq E$ denotes the set of all edges that are incident to a node $i \in V$. Moreover, a number $k \in \{1, 2, \ldots, n\}$ is given.

The goal is to choose exactly k nodes from V, which we call *exemplars*, and to connect each other node, called *non-exemplar*, to exactly one neighboring exemplar such that the total costs of the used edges are minimized. We therefore define the following two sets of binary decision variables:

- $y_i \in \{0,1\}$, $i \in V$: Indicates if node i is chosen as an exemplar ($y_i = 1$) or not ($y_i = 0$).
- $x_e \in \{0,1\}$, $e = (i,j) \in E$: Indicates if nodes i and j should be connected ($x_e = 1$) or not ($x_e = 0$). The costs of this connection are given by $c(e)$.

Figure 1 shows an example instance and a respective solution for $n = 8$ and $k = 3$. Note that there may be exemplars with no nodes connected to them, e.g., node 3 in the figure.

Model this problem as a MILP. Your model should address the following aspects.

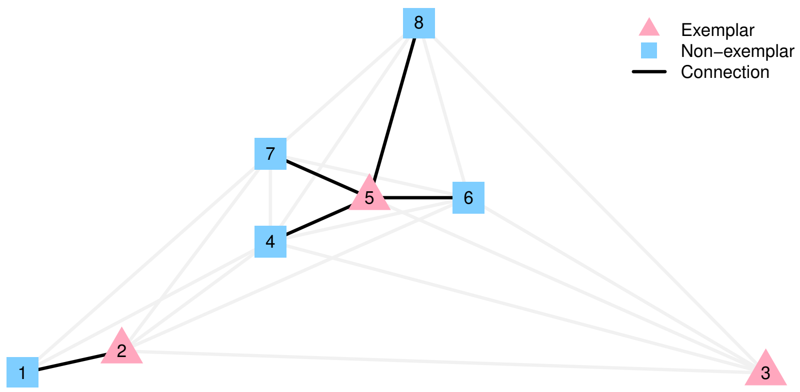


Figure 1: Example instance for $n = 8$ and $k = 3$.

c) (1 Point) Formulate the objective function.

d) (2 Points) Make sure that the right number of nodes are selected as exemplars and that the appropriate number of edges (connections) are chosen.

e) (2 Points) Make sure that each selected edge connects exactly one exemplar with one non-exemplar, i.e., no two exemplars and no two non-exemplars are connected.

f) (2 Points) Moreover, make sure that each non-exemplar is connected to exactly one exemplar.

There are several possibilities for modelling this problem. If a requirement from d), e) or f) is not obviously satisfied by a constraint, then briefly argue why your constraints are sufficient.

Question 6: Geometric Algorithms **(5 Points)**

Let P be a set of n points in the plane $\mathbb{R}^2$. You can assume that no four points lie on a common circle and that no two pairwise distances within P are equal. We define for each point $p \in P$ its unique *nearest neighbor* $\alpha(p) \in P$, i.e., $d(p, \alpha(p)) \leq d(p, q)$ for all $q \in P \setminus \{p\}$, where $d(p, q)$ denotes the Euclidean distance between two points p and q.

a) Let $\mathcal{DG}(P)$ be the Delaunay graph of P. Prove that for each point $p \in P$ the pair $(p, \alpha(p))$ defines an edge in $\mathcal{DG}(P)$.

b) Assume you are given the Delaunay graph $\mathcal{DG}(P)$ as input. Sketch a linear-time algorithm that computes the nearest neighbor $\alpha(p)$ for all points $p \in P$. Argue why it runs in $O(n)$ time.

   Hint: Use the property from (a).