VU Discrete Mathematics

Exercises for 11th November 2025

25) Let $\mathcal{T}$ be the class of rooted and labelled trees, i.e. the n vertices of a tree of size n are labelled with the labels $1, 2, \dots, n$. Use the theory of combinatorial constructions to determine a functional equation for the exponential generating function of $\mathcal{T}$. Finally, apply the following theorem to prove that the number of trees in $\mathcal{T}$ which have n vertices is equal to n^{n-1}. (You are not asked to prove that theorem.)

Theorem. *Let $\Phi(w) = \sum_{n\geq 0} \phi_n w^n$ with $\phi_0 \neq 0$. If $z = w/\Phi(w)$, then $[z^n]w = \frac{1}{n}[w^{n-1}]\Phi(w)^n$.*

26–27) **Note: Do not refer to integer factorization in your proofs!**

26) Let a, b, c, d be integers. Prove:

a) If $a \mid b$ and $a \mid c$, then for all integers x, y we have $a \mid (xb + yc)$.

b) If $\gcd(a, b) = 1$ and $c \mid a$ and $d \mid b$, then $\gcd(c, d) = 1$.

c) If $a \mid c$ and $b \mid c$ and $\gcd(a, b) = 1$, then $ab \mid c$

27) Prove: If $\gcd(a, b) = 1$ then $\gcd(a + b, a - b)$ is either 1 or 2.

28) Prove that the equation $w^2 + x^2 - 3y^2 - 3z^2 = 0$ has no solution over $\mathbb{N} \setminus \{0\}$.

29) Use the Euclidean algorithm to find all greatest common divisors of $x^3 + 5x^2 + 7x + 3$ and $x^3 + x^2 - 5x + 3$ in $\mathbb{Q}[x]$.

30) Prove that in a commutative ring with 1 for all elements a and b and all units x we always have $\gcd(a, b) = \gcd(ax, b)$.